Filomat 29:3 (2015), 381–392
DOI 10.2298/FIL1503381B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Timelike Rotational Surfaces of Elliptic, Hyperbolic and Parabolic Types in Minkowski Space $\mathbb{E}^4_1$ with Pointwise 1-Type Gauss Map

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Abstract. In this work, we focus on a class of timelike rotational surfaces in Minkowski space $\mathbb{E}^4_1$ with 2-dimensional axis. There are three types of rotational surfaces with 2-dimensional axis, called rotational surfaces of elliptic, hyperbolic or parabolic type. We obtain all flat timelike rotational surface of elliptic and hyperbolic types with pointwise 1-type Gauss map of the first and second kind. We also prove that there exists no flat timelike rotational surface of parabolic type in $\mathbb{E}^4_1$ with pointwise 1-type Gauss map.

1. Introduction

The notion of finite type submanifolds of Euclidean space was introduced by B.-Y. Chen in late 1970's,[3]. Since then many geometers have done works to characterize or classify the submanifolds of Euclidean and pseudo-Euclidean space in terms of finite type. In [5], the definiton of finite type was given similarly for differentiable maps, in particular, to Gauss map of submanifolds. A smooth map ϕ on a submanifold M of a Euclidean space or a pseudo-Euclidean space is said to be of *finite type* if ϕ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $\phi = \phi_0 + \sum_{i=1}^{k} \phi_i$, where ϕ_0 is a constant map, $\phi_1, \ldots, \phi_k$ nonconstant maps such that $\Delta\phi_i = \lambda_i\phi_i,\ \lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then ϕ is said to be of k-type. From these definition, one can immediately seen that a submanifold M of a Euclidean space or a pseudo Euclidean space has 1-type Gauss map if and only if its Gauss map ν satisfies $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and for some constant vector C. In addition, B.-Y. Chen and P. Piccinni characterized compact submanifolds of Euclidean space with finite type Gauss map, [5]. Besides this, there are many articles on submanifolds with finite type Gauss map. (cf. [2, 4, 6])

In time, it has been observed that there are some geometrically important submanifolds, such as helicoids, B-scrolls in a 3-dimensional Minkowski space $\mathbb{E}^3_1$, generalized catenoids, spherical n-cones and Enneper's hypersurfaces in $\mathbb{E}^{n+1}_1$ whose Gauss map satisfies

$$\Delta\nu = f(\nu + C) \tag{1}$$

for some smooth function f on M and some constant vector C ([9, 14]). A submanifold of a Euclidean space or a pseudo Euclidean space is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for

2010 *Mathematics Subject Classification*. Primary 53B25; Secondary 53C50
Keywords. Pointwise 1-type Gauss map, rotational surface, timelike surface, flat surface, Minkowski space
Received: 17 July 2014; Accepted: 21 October 2014
Communicated by Ljubica Velimirović and Mića Stanković
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some smooth function f on M and some constant vector C. In particular, C is zero, it is said to be *the first kind*. Otherwise, it is said to be of *the second kind*. (cf. [1, 7, 13])

Furthermore, rotational surfaces in a Euclidean space and a pseudo Euclidean space with pointwise 1-type Gauss map were recently studied. The complete classification of ruled surfaces in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the first kind was obtained in [14]. Ruled surfaces in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind were studied in [8, 11]. Also, a complete classification of rational surfaces of revolution in $\mathbb{E}^3_1$ satisfying (1) was given in [13], and it was proved that a right circular cone and a hyperbolic cone in $\mathbb{E}^3_1$ are the only rational surfaces of revolution in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind. The second author and Turgay studied general rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map, [12]. Moreover, in [15] a complete classification of cylindrical and non-cylindrical surfaces in $\mathbb{E}^m_1$ with pointwise 1-type Gauss map of the first kind was obtained. Recently, the authors studied rotational spacelike surfaces of elliptic, hyperbolic and parabolic types in $\mathbb{E}^4_1$, [10].

In this article, we focus on a class of timelike rotational surfaces in Minkowski space $\mathbb{E}^4_1$ with 2-dimensional axis. There are three types of rotational surfaces with 2-dimensional axis, called rotational surfaces of elliptic, hyperbolic or parabolic type which are invariant under spacelike rotation, hyperbolic rotation and screw rotation, respectively.

We obtain all flat timelike rotational surface of elliptic and hyperbolic types with pointwise 1-type Gauss map of the first and second kind. We also show that there exists no flat timelike rotational surface of parabolic type in $\mathbb{E}^4_1$ with pointwise 1-type Gauss map.

2. Prelimineries

Let $\mathbb{E}^m_1$ denote m-dimensional Minkowski space with the canonical metric given by

$$g = dx_1^2 + dx_2^2 + \cdots + dx_{m-1}^2 - dx_m^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m_1$.

A vector $v \in \mathbb{E}^m_1$ is called spacelike (resp.,timelike) if $\langle v, v\rangle > 0$ or $v = 0$ (resp., $\langle v, v\rangle < 0$). A vector v is called lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$.

Let M be an oriented n-dimensional pseudo Riemannian submanifold in an $(n+2)$-dimensional Minkowski space $\mathbb{E}^{n+2}_1$. We choose local orthonormal frame $\{e_1, \ldots, e_{n+2}\}$ on M with $\varepsilon_A = \langle e_A, e_A\rangle = \pm 1$ such that $e_1, \ldots, e_n$ are tangent to M and e_{n+1}, e_{n+2} are normal to M. We use the following convention on the range of indices: $1 \leq i, j, k, \ldots \leq n,\ n+1 \leq r, s, t, \ldots \leq n+2$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^{n+2}_1$ and ∇ the induced connection on M. Denote by $\{\omega^1, \ldots, \omega^{n+2}\}$ the dual frame and by $\{\omega_{AB}\}, A, B = 1, \ldots, n+2$, the connection 1-forms associated to $\{e_1, \ldots, e_{n+2}\}$ with $\omega_{AB} + \omega_{BA} = 0$. Then the Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{n+2} \varepsilon_r h^r_{ik} e_r$$

and

$$\widetilde{\nabla}_{e_k} e_r = -A_r(e_k) + D_{e_k} e_r, \qquad D_{e_k} e_r = \sum_{s=n+1}^{n+2} \varepsilon_s \omega_{rs}(e_k) e_s$$

where D is the normal connection, h^r_{ij} the coefficients of the second fundamental form h, and A_r the Weingarten map in the direction e_r.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n} \sum_{i,r} \varepsilon_i \varepsilon_r h^r_{ii} e_r \tag{2}$$

and

$$||h||^2 = \sum_{i,j,r} \varepsilon_i \varepsilon_j \varepsilon_r h^r_{ij} h^r_{ji}. \tag{3}$$

A submanifold M is said to have parallel mean curvature vector if the mean curvature vector satisfies $DH = 0$ identically. Also, the Gaussian curvature K is given by

$$K = \sum_{s=n+1}^{n+2} \varepsilon_s (h^s_{11} h^s_{22} - h^s_{12} h^s_{21}). \tag{4}$$

If K vanishes identically, M is said to be flat.

The Codazzi equation of M in $\mathbb{E}^{n+2}_1$ is given by

$$\begin{aligned} h^r_{ij,k} &= h^r_{jk,i}, \\ h^r_{jk,i} &= e_i(h^r_{jk}) + \sum_{s=n+1}^{n+2} \varepsilon_s h^s_{jk} \omega_{sr}(e_i) - \sum_{\ell=1}^{n} \varepsilon_\ell \left(\omega_{j\ell}(e_i) h^r_{\ell k} + \omega_{k\ell}(e_i) h^r_{\ell j} \right). \end{aligned} \tag{5}$$

Also, from the Ricci equation of M in $\mathbb{E}^{n+2}_1$, we get

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k \rangle = \sum_{i=1}^{n} \varepsilon_i \left(h^r_{ik} h^s_{ij} - h^r_{ij} h^s_{ik} \right), \tag{6}$$

where R^D is the normal curvature tensor.

A submanifold M in $\mathbb{E}^{n+2}_1$ is said to have flat normal bundle if the normal curvature tensor R^D vanishes identically.

The gradient of a smooth function f defined on M into $\mathbb{R}$ is defined by $\nabla f = \sum_{i=1}^{n} \varepsilon_i e_i(f) e_i$, and the Laplace operator of M with respect to induced metric is $\Delta = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)$.

Let $G(m-n, m)$ be the Grassmannian manifold consisting of all oriented $(m-n)$–planes through the origin of $\mathbb{E}^m_t$ and $\bigwedge^{m-n} \mathbb{E}^m_t$ the vector space obtained by the exterior product of $m-n$ vectors in $\mathbb{E}^m_t$. Let $f_{i_1} \wedge \cdots \wedge f_{i_{m-n}}$ and $g_{i_1} \wedge \cdots \wedge g_{i_{m-n}}$ be two vectors in $\bigwedge^{m-n} \mathbb{E}^m_t$, where $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_m\}$ are two orthonormal bases of $\mathbb{E}^m_t$. Define an indefinite inner product $\langle , \rangle$ on $\bigwedge^{m-n} \mathbb{E}^m_t$ by

$$\langle f_{i_1} \wedge \cdots \wedge f_{i_{m-n}}, g_{i_1} \wedge \cdots \wedge g_{i_{m-n}} \rangle = \det(\langle f_{i_\ell}, g_{j_k} \rangle). \tag{7}$$

Therefore, for some positive integer s, we may identify $\bigwedge^{m-n} \mathbb{E}^m_t$ with some pseudo-Euclidean space $\mathbb{E}^N_s$, where $N = \binom{m}{m-n}$. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an oriented local orthonormal frame on an n-dimensional pseudo-Riemannian submanifold M in $\mathbb{E}^m_t$ with $\varepsilon_B = \langle e_B, e_B \rangle = \pm 1$ such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M. The map $\nu : M \to G(m-n, m) \subset \mathbb{E}^N_s$ from an oriented pseudo-Riemannian submanifold M into $G(m-n, m)$ defined by

$$\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p) \tag{8}$$

is called the *Gauss map* of M that is a smooth map which assigns to a point p in M the oriented $(m-n)$–plane through the origin of $\mathbb{E}^m_t$ and parallel to the normal space of M at p, [16].

We put $\varepsilon = \langle \nu, \nu \rangle = \varepsilon_{n+1} \varepsilon_{n+2} \cdots \varepsilon_m = \pm 1$ and

$$\widetilde{M}^{N-1}_s(\varepsilon) = \begin{cases} \mathbb{S}^{N-1}_s(1) & \text{in } \mathbb{E}^N_s \text{ if } \varepsilon = 1 \\ \mathbb{H}^{N-1}_{s-1}(-1) & \text{in } \mathbb{E}^N_s \text{ if } \varepsilon = -1. \end{cases}$$

Then the Gauss image $\nu(M)$ can be viewed as $\nu(M) \subset \widetilde{M}^{N-1}_s(\varepsilon)$.

2.1. Rotational surfaces in $\mathbb{E}^4_1$

In this work, we consider a class of rotational surfaces in Minkowski space $\mathbb{E}^4_1$ with 2-dimensional axis. There are 3-types of rotational surfaces with 2-dimensional axis, called rotational surfaces of elliptic, hyperbolic or parabolic type. These surfaces are invariant under spacelike rotation, hyperbolic rotation and screw rotation, respectively.

Rotational Surfaces of Elliptic Type:
Let $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ be the standard orthonormal basis of $\mathbb{E}^4_1$, i.e., $\eta_1 = (1, 0, 0, 0)$, $\eta_2 = (0, 1, 0, 0)$, $\eta_3 = (0, 0, 1, 0)$, $\eta_4 = (0, 0, 0, 1)$. Let $\alpha : I \to \mathbb{E}^4_1$, $\alpha(s) = (x(s), 0, z(s), w(s))$ be a smooth regular curve defined on an open interval $I \subseteq \mathbb{R}$. The curve $\alpha(s)$ lies in the 3-dimensional subspace $\mathbb{E}^3_1 = \text{span}\{\eta_1, \eta_3, \eta_4\}$ of $\mathbb{E}^4_1$. We consider the surface defined by

$$M_1 : F_1(s, t) = (x(s) \cos t, x(s) \sin t, z(s), w(s)), \quad s \in I,\ t \in [0, 2\pi), \tag{9}$$

which is the orbit of $\alpha(s)$ under the action of the orthogonal transformation of $\mathbb{E}^4_1$ that leaves the timelike plane spanned by $\{\eta_3, \eta_4\}$ pointwise fixed. The surface M_1 is called a rotational surface of elliptic type, and it is regular and timelike if $x(s) > 0$ and the profile curve $\alpha(s)$ is timelike on I.

Rotational Surfaces of Hyperbolic Type:
Let $\beta : I \to \mathbb{E}^4_1$, $\beta(s) = (x(s), y(s), 0, w(s))$ be a smooth regular curve defined on an open interval $I \subseteq \mathbb{R}$. The curve $\beta(s)$ lies in the 3-dimensional subspace $\mathbb{E}^3_1 = \text{span}\{\eta_1, \eta_2, \eta_4\}$ of $\mathbb{E}^4_1$. Now we consider the surface defined by

$$M_2 : F_2(s, t) = (x(s), y(s), w(s) \sinh t, w(s) \cosh t), \quad s \in I, t \in \mathbb{R}, \tag{10}$$

which is the orbit of $\beta(s)$ under the action of the orthogonal transformation of $\mathbb{E}^4_1$ that leaves the spacelike plane spanned by $\{\eta_1, \eta_2\}$ pointwise fixed. The surface M_2 is called a rotational surface of hyperbolic type, and it is regular and timelike if $w(s) > 0$ and the profile curve $\beta(s)$ is timelike on I.

Rotational Surfaces of Parabolic Type:
Now we take the pseudo-orthonormal basis $\{\eta_1, \eta_2, \xi_3, \xi_4\}$ of $\mathbb{E}^4_1$ such that $\xi_3 = (\eta_4 - \eta_3)/\sqrt{2}$, $\xi_4 = (\eta_3 + \eta_4)/\sqrt{2}$. Note that $\langle \xi_3, \xi_3 \rangle = 0$, $\langle \xi_4, \xi_4 \rangle = 0$, $\langle \xi_3, \xi_4 \rangle = -1$. We consider a smooth regular curve $\gamma(s) = x(s)\eta_1 + \bar{z}(s)\eta_3 + \bar{w}(s)\eta_4$, $s \in I$ lying in the subspace $\mathbb{E}^3_1 = \text{span}\{\eta_1, \eta_3, \eta_4\}$. Since $\eta_3 = (\xi_4 - \xi_3)/\sqrt{2}$ and $\eta_4 = (\xi_3 + \xi_4)/\sqrt{2}$, we have

$$\gamma(s) = x(s)\eta_1 + \Big(\frac{\bar{w}(s) - \bar{z}(s)}{\sqrt{2}}\Big)\xi_3 + \Big(\frac{\bar{w}(s) + \bar{z}(s)}{\sqrt{2}}\Big)\xi_4, \quad s \in I. \tag{11}$$

If we put $z(s) = (\bar{w}(s) - \bar{z}(s))/\sqrt{2}$ and $w(s) = (\bar{w}(s) + \bar{z}(s))/\sqrt{2}$, then we have $\gamma(s) = x(s)\eta_1 + z(s)\xi_3 + w(s)\xi_4$. Now we consider the surface defined by

$$M_3 : F_3(s, t) = x(s)\eta_1 + \sqrt{2}tw(s)\eta_2 + (z(s) + t^2 w(s))\xi_3 + w(s)\xi_4, \quad s \in I, t \in \mathbb{R}, \tag{12}$$

which is the orbit of $\gamma(s)$ under the action of the orthogonal transformation T of $\mathbb{E}^4_1$ defined by $T(\eta_1) = \eta_1$, $T(\eta_2) = \eta_2 + \sqrt{2}t\xi_3$, $T(\xi_3) = \xi_3$ and $T(\xi_4) = \sqrt{2}t\eta_2 + t^2\xi_3 + \xi_4$ that leaves the degenerate plane spanned by $\{\eta_1, \xi_3\}$ pointwise fixed. The surface M_3 is screw invariant and also called a rotational surface of parabolic type. It is regular and timelike if $w(s) > 0$ and the profile curve $\gamma(s)$ is timelike on I.

3. Timelike rotational surfaces with pointwise 1-type Gauss map

In this section, we study timelike rotational surfaces of elliptic, hyperbolic or parabolic type with pointwise 1-type Gauss map. There are flat timelike rotational surfaces of elliptic and hyperbolic type in $\mathbb{E}^4_1$ with pointwise 1-type Gauss map of the first and second kind. However, for flat timelike rotational surfaces of parabolic type in $\mathbb{E}^4_1$, the Gauss map ν does not satisfy (1), that is, it is not of pointwise 1-type Gauss map.

The Laplacian of the Gauss map ν for an n-dimensional submanifold M in a pseudo-Euclidean space $\mathbb{E}^{n+2}_t$ was given by

Lemma 3.1. *[12] Let M be an n-dimensional submanifold of a pseudo-Euclidean space $\mathbb{E}_t^{n+2}$. Then, the Laplacian of the Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned}\Delta\nu =& \|h\|^2\nu + 2\sum_{j<k} \varepsilon_j\varepsilon_k R^D(e_j, e_k; e_{n+1}, e_{n+2})e_j \wedge e_k + \nabla(\mathrm{tr}A_{n+1}) \wedge e_{n+2} + e_{n+1} \wedge \nabla(\mathrm{tr}A_{n+2}) \\ &+ n\sum_{j=1}^{n} \varepsilon_j\omega_{(n+1)(n+2)}(e_j)H \wedge e_j,\end{aligned} \tag{13}$$

where $\|h\|^2$ is the squared length of the second fundamental form, R^D the normal curvature tensor, and $\nabla(\mathrm{tr}A_r)$ the gradient of $\mathrm{tr}A_r$.

Let M be a timelike surface in $\mathbb{E}_1^4$. We choose a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ defined on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M. Let C be a vector field in $\Lambda^2\mathbb{E}_1^4 \equiv \mathbb{E}_3^6$. Since the set $\{e_A \wedge e_B | 1 \le A < B \le 4\}$ is an orthonormal basis for $\mathbb{E}_3^6$, the vector C can be expressed as

$$C = \sum_{1\le A<B\le 4} \varepsilon_A\varepsilon_B C_{AB}\, e_A \wedge e_B, \tag{14}$$

where $C_{AB} = \langle C, e_A \wedge e_B\rangle$. Let e_1 be timelike, we have $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$.

Then the condition of the constancy of the vector C written by (14) is similar to the constancy of C on a spacelike surface in $\mathbb{E}_1^4$, given in [12]:

Lemma 3.2. *A vector C in $\Lambda^2\mathbb{E}_1^4 \equiv \mathbb{E}_3^6$ written by* (14) *is constant if and only if the following equations are satisfied for $i = 1, 2$*

$$e_i(C_{12}) = h_{i2}^3 C_{13} + h_{i2}^4 C_{14} - h_{i1}^3 C_{23} - h_{i1}^4 C_{24}, \tag{15}$$

$$e_i(C_{13}) = -h_{i2}^3 C_{12} + \omega_{34}(e_i)C_{14} + \omega_{12}(e_i)C_{23} - h_{i1}^4 C_{34}, \tag{16}$$

$$e_i(C_{14}) = -h_{i2}^4 C_{12} - \omega_{34}(e_i)C_{13} + \omega_{12}(e_i)C_{24} + h_{i1}^3 C_{34}, \tag{17}$$

$$e_i(C_{23}) = -h_{i1}^3 C_{12} + \omega_{12}(e_i)C_{13} + \omega_{34}(e_i)C_{24} - h_{i2}^4 C_{34}, \tag{18}$$

$$e_i(C_{24}) = -h_{i1}^4 C_{12} + \omega_{12}(e_i)C_{14} - \omega_{34}(e_i)C_{23} + h_{i2}^3 C_{34}, \tag{19}$$

$$e_i(C_{34}) = -h_{i1}^4 C_{13} + h_{i1}^3 C_{14} + h_{i2}^4 C_{23} - h_{i2}^3 C_{24}. \tag{20}$$

3.1. Rotational surfaces of elliptic type with pointwise 1-type Gauss Map

In this subsection, we determine flat timelike rotational surfaces of elliptic type in $\mathbb{E}_1^4$ with pointwise 1-type Gauss map of the first and second kind.

Let M_1 be a timelike rotational surface of elliptic type in $\mathbb{E}_1^4$ defined by (9). Without loss of generality, we assume that the smooth timelike profile curve α is parametrized by its arc lenght s, i.e., $x'^2(s) + z'^2(s) - w'^2(s) = -1$.

We also suppose that the principal direction α'' is nonnull. Then the curvature function κ of α is given by $\kappa(s) = \sqrt{\langle\alpha''(s), \alpha''(s)\rangle} = \sqrt{x''^2(s) + z''^2(s) - w''^2(s)} \neq 0,\ s \in I$.

For the surface M_1, we choose a moving frame field $\{e_1, e_2, e_3, e_4\}$ such that e_1, e_2 are tangent to M_1, and e_3, e_4 are normal to M_1 which are given as follows:

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{x(s)}\frac{\partial}{\partial t}, \quad x(s) > 0, \tag{21}$$

$$e_3 = \frac{1}{\kappa}(x''\cos t, x''\sin t, z'', w''), \tag{22}$$

$$e_4 = \frac{1}{\kappa}(\mu\cos t, \mu\sin t, w'x'' - x'w'', z'x'' - x'z''), \tag{23}$$

where $\mu = z'w'' - w'z''$, $\varepsilon_1 = \langle e_1, e_1\rangle = -1$, $\varepsilon_2 = \langle e_2, e_2\rangle = 1$, $\varepsilon_3 = \langle e_3, e_3\rangle = 1$ and $\varepsilon_4 = \langle e_4, e_4\rangle = 1$.

By a direct computation, we can obtain the coefficients of the second fundamental form and the connection forms as follows

$$h^3_{11} = \kappa, \qquad h^3_{22} = -\frac{x''}{\kappa x}, \quad h^3_{12} = 0, \tag{24}$$

$$h^4_{11} = h^4_{12} = 0, \qquad h^4_{22} = -\frac{\mu}{\kappa x}, \tag{25}$$

$$\omega_{12}(e_1) = 0, \qquad \omega_{12}(e_2) = \frac{x'}{x}, \tag{26}$$

$$\omega_{34}(e_1) = \tau, \qquad \omega_{34}(e_2) = 0, \tag{27}$$

where τ is the torsion of the profile curve α. Therefore, we obtain the mean curvature vector and the Gaussian curvature of M_1, respectively, as follows

$$H = -\frac{1}{2}\Big[\Big(\frac{x''}{\kappa x} + \kappa\Big)e_3 + \frac{\mu}{\kappa x}e_4\Big] \quad \text{and} \quad K = -\frac{x''}{x}. \tag{28}$$

On the other hand, by using the Codazzi equation (5) we get

$$e_1(h^3_{22}) = -\omega_{12}(e_2)\left(h^3_{11} + h^3_{22}\right) + \tau h^4_{22}, \tag{29}$$

$$e_1(h^4_{22}) = -\omega_{12}(e_2)h^4_{22} - \tau h^3_{22}. \tag{30}$$

Theorem 3.3. *Let M_1 be a flat timelike rotational surface of elliptic type in $\mathbb{E}^4_1$ defined by* (9). *If the profile curve $\alpha(s) = (x(s), 0, z(s), w(s))$ has the nonnull principal curvature vector $\alpha''(s)$, then*

i. M_1 has global 1-type Gauss map of the first kind if and only if $\alpha(s)$ is given by

$$x(s) = x_1,$$

$$z(s) = \frac{1}{q_0}\cosh(\kappa_1 + q_0 s) + z_0, \tag{31}$$

$$w(s) = \frac{1}{q_0}\sinh(\kappa_1 + q_0 s) + w_0, \tag{32}$$

where $q_0 = \pm\kappa_0$ and $x_1, z_0, w_0, \kappa_0, \kappa_1 \in \mathbb{R}$ with $x_1, \kappa_0 > 0$. Moreover, $\Delta\nu = \Big(\frac{1}{x_1^2} + \kappa_0^2\Big)\nu$.

ii. M_1 has pointwise 1-type Gauss map of the second kind if and only if $\alpha(s)$ is given by

$$x(s) = x_0 s + x_1,$$

$$z(s) = \sqrt{x_0^2 + 1}\int \sinh(q_0 \ln(x_0 s + x_1) + \psi_0)ds + z_0, \tag{33}$$

$$w(s) = \sqrt{x_0^2 + 1}\int \cosh(q_0 \ln(x_0 s + x_1) + \psi_0)ds + w_0, \tag{34}$$

and the Gauss map $\nu = e_3 \wedge e_4$ satisfies (1) *for the function*

$$f(s,t) = \frac{\kappa_0^2 + x_0^2 + 1}{(x_0^2 + 1)(x_0 s + x_1)^2}$$

and for the constant vector

$$C = x_0^2 e_3 \wedge e_4 - \frac{q_0 x_0^2(x_0^2 + 1)}{\kappa_0} e_1 \wedge e_3$$

where $s > -x_1/x_0$, $q_0 = \pm\frac{\kappa_0}{x_0\sqrt{x_0^2+1}}$ and $\kappa_0, x_0, x_1, z_0, w_0, \psi_0 \in \mathbb{R}$ with $\kappa_0 > 0$. The integrals given above can be evaluated according to $q_0 \neq \pm 1, q_0 = 1$ or $q_0 = -1$. Moreover, the profile curve α is a helix.

Proof. Let M_1 be a timelike rotational surface in $\mathbb{E}^4_1$ defined by (9). Suppose that the principal direction $\alpha''(s)$ is nonnull. Then we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M_1 in $\mathbb{E}^4_1$ given by (21)–(23), and the entries of the shape operators A_3 and A_4 are given by (24) and (25), respectively. Hence the Laplacian of the Gauss map $\nu = e_3 \wedge e_4$ from (13) is obtained as

$$\Delta\nu = \|h\|^2\nu + (e_1(h^4_{22}) + \tau(h^3_{22} - h^3_{11}))e_1 \wedge e_3 + (e_1(h^3_{11} - h^3_{22}) + \tau h^4_{22})e_1 \wedge e_4.$$

If we use the Codazzi equations (29) and (30), then we get

$$\Delta\nu = \|h\|^2\nu - (\tau h^3_{11} + \omega_{12}(e_2)h^4_{22})e_1 \wedge e_3 + (\kappa' + \omega_{12}(e_2)(h^3_{11} + h^3_{22}))e_1 \wedge e_4. \tag{35}$$

Suppose that M_1 is flat and has pointwise 1-type Gauss map of the second kind. Then, from the second equation of (28) we have $x'' = 0$, that is, $x(s) = x_0 s + x_1$ with $x_0, x_1 \in \mathbb{R}$, and hence $h^3_{22} = 0$. Considering (35) and $h^3_{22} = 0$, we obtain

$$f(1 + C_{34}) = \|h\|^2 = (h^3_{11})^2 + (h^4_{22})^2, \tag{36}$$

$$fC_{13} = \tau h^3_{11} + \omega_{12}(e_2)h^4_{22}, \tag{37}$$

$$fC_{14} = -\kappa' - \omega_{12}(e_2)h^3_{11}, \tag{38}$$

$$C_{12} = C_{23} = C_{24} = 0. \tag{39}$$

For $i = 2$, from (15), (18) and (19) we have, respectively,

$$h^4_{22}C_{14} = 0, \tag{40}$$

$$\omega_{12}(e_2)C_{13} - h^4_{22}C_{34} = 0, \tag{41}$$

$$\omega_{12}(e_2)C_{14} = 0. \tag{42}$$

From (26), we have $\omega_{12}(e_2) = \frac{x_0}{x}$, and thus (42) implies that $x_0C_{14} = 0$ from which $x_0 = 0$ or $C_{14} = 0$.

CASE I. $x_0 = 0$. So, $x(s) = x_1$ and $x_1 \in \mathbb{R}_+$ as $x(s) > 0$. In this case, the profile curve $\alpha(s) = (x_1, 0, z(s), w(s))$ is a plane curve, that is, $\tau(s) = 0$, and $\langle\alpha'(s), \alpha'(s)\rangle = z'^2(s) - w'^2(s) = -1$. Now we may put $z'(s) = \sinh\theta(s)$, $w'(s) = \cosh\theta(s)$. Then $\alpha''(s) = (0, 0, \theta'\cosh\theta, \theta'\sinh\theta)$ and $\langle\alpha'', \alpha''\rangle = \theta'^2$. This means that $\kappa = |\theta'|$. So, $\theta'(s) = \pm\kappa(s)$. On the other hand, the curvature of $\alpha(s)$ is also given by $\kappa = |z'w'' - w'z''| = |\mu|$, that is, $\mu = \pm\kappa$. Therefore, from (25), (26) and (27) we have, respectively, $h^4_{22} = \pm\frac{1}{x_1}$, $\omega_{12} = 0$ and $\omega_{34} = 0$. Now, considering these quantities, (40) implies that $C_{14} = 0$. Hence, (38) gives $\kappa = \kappa_0$, where $\kappa_0 \in \mathbb{R}_+$. So, we have $\theta(s) = q_0 s + \kappa_1$ where $q_0 = \pm\kappa_0$. Also, (37) implies that $C_{13} = 0$, and thus (41) produces $C_{34} = 0$. Therefore, $C = 0$, namely, the Gauss map ν is not of pointwise 1-type of the second kind. Moreover, (36) gives $f(s,t) = \frac{1}{x_1^2} + \kappa_0^2$ which is a nonzero constant, and ν is of global 1-type of the first kind. As $\theta(s) = q_0 s + \kappa_1$, from $z'(s) = \sinh\theta(s)$ and $w'(s) = \cosh\theta(s)$ we have (31) and (32).

CASE II. $C_{14} = 0$ and $x_0 \neq 0$. So, we get $\kappa' + \frac{x_0}{x}\kappa = 0$ from (38) which yields $\kappa = \frac{\kappa_0}{x} = \frac{\kappa_0}{x_0 s + x_1}$ for some constant $\kappa_0 > 0$. Now, since α is parametrized by arc length parameter s, we get $z'^2 - w'^2 = -1 - x_0^2$. Say $\mu_0^2 = x_0^2 + 1 > 0$. Then $z'^2 - w'^2 = -\mu_0^2$. Without loss of generality, we assume that $\mu_0 > 0$. When we write $w'(s) = \mu_0\cosh\psi(s)$ and $z'(s) = \mu_0\sinh\psi(s)$, we get $w''(s) = \mu_0\psi'(s)\sinh\psi(s)$ and $z''(s) = \mu_0\psi'(s)\cosh\psi(s)$. Thus, $\kappa^2 = \langle\alpha'', \alpha''\rangle = z''^2 - w''^2 = \mu_0^2\psi'^2$ from which we obtain $\psi(s) = q_0\ln(x_0 s + x_1) + \psi_0$, where $q_0 = \pm\frac{\kappa_0}{x_0\sqrt{x_0^2+1}}$ and ψ_0 is an integration constant. Therefore, we obtain (33) and (34).

Also, we have $\mu = z'w'' - w'z'' = -\mu_0^2\psi' = -\mu_0^2\frac{q_0x_0}{x}$. Hence, $h^4_{22} = -\frac{\mu}{\kappa x} = \mu_0^2\frac{q_0x_0}{\kappa_0 x}$ and equation (41) gives

$$\kappa_0C_{13} - \mu_0^2q_0C_{34} = 0. \tag{43}$$

Now, by a calculation we have $\tau h^3_{11} + \omega_{12}(e_2)h^4_{22} = \dfrac{q_0x_0^2(\mu_0^2+\kappa_0^2)}{\kappa_0 x^2}$ and $\|h\|^2 = (h^3_{11})^2 + (h^4_{22})^2 = \dfrac{\kappa_0^2+\mu_0^2}{x^2}$. From (36) and (37), we get $\kappa_0 C_{13} - q_0x_0^2C_{34} = q_0x_0^2$ from which and (43) we obtain that $C_{13} = \dfrac{x_0^2\mu_0^2q_0}{\kappa_0}$ and $C_{34} = x_0^2$. Hence, we get $C = x_0^2 e_3\wedge e_4 - q_0\dfrac{x_0^2(x_0^2+1)}{\kappa_0}e_1\wedge e_3$ by (14). Also, from equation (36) we have $f(s,t) = \dfrac{x_0^2+\kappa_0^2+1}{(x_0^2+1)(x_0s+x_1)^2}$. Considering the curve $\alpha(s)$ lying in $\mathbb{E}^3_1$, we obtain $\tau = \dfrac{\langle \alpha'\times\alpha'', \alpha'''\rangle}{\kappa^2} = \dfrac{q_0x_0^2}{x}$ by a direct computation. Note that τ/κ is a constant, and thus the profile curve α is a helix.

The converse of the proof follows from a direct calculation. □

Note that if $\alpha''(s) = 0$ which means $\alpha(s)$ is a line, then it can be shown that M_1 lies in Euclidean space $\mathbb{E}^3$ or Minkowski space $\mathbb{E}^3_1$. Rotational surfaces in Euclidean space $\mathbb{E}^3$ or Minkowski space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map were studied in [7, 13].

3.2. Rotational surfaces of hyperbolic type with pointwise 1-type Gauss Map

In this subsection, we study flat timelike rotational surfaces of hyperbolic type in $\mathbb{E}^4_1$ with pointwise 1-type Gauss map of the second kind.

Let M_2 be a timelike rotational surface of hyperbolic type in $\mathbb{E}^4_1$ defined by (10). Without loss of generality, we assume that the smooth timelike profile curve β is parametrized by its arc lenght s, i.e., $x'^2(s) + y'^2(s) - w'^2(s) = -1$. We also suppose that the principal direction β'' is nonnull. Then the curvature function κ of β is given by $\kappa(s) = \sqrt{\langle\beta''(s), \beta''(s)\rangle} = \sqrt{x''^2(s) + y''^2(s) - w''^2(s)} \neq 0,\ s\in I$.

For the surface M_2, we choose a moving frame field $\{e_1, e_2, e_3, e_4\}$ such that e_1, e_2 are tangent to M_2, and e_3, e_4 are normal to M_2 which are given as follows:

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{w(s)}\frac{\partial}{\partial t}, \quad w(s) > 0, \tag{44}$$

$$e_3 = \frac{1}{\kappa}(x'', y'', w''\sinh t, w''\cosh t), \tag{45}$$

$$e_4 = \frac{1}{\kappa}(y'w'' - w'y'', w'x'' - x'w'', \rho\sinh t, \rho\cosh t), \tag{46}$$

where $\rho = y'x'' - x'y''$, $\varepsilon_1 = \langle e_1, e_1\rangle = -1$, $\varepsilon_2 = \langle e_2, e_2\rangle = 1$, $\varepsilon_3 = \langle e_3, e_3\rangle = 1$ and $\varepsilon_4 = \langle e_4, e_4\rangle = 1$.

By a direct computation, we can obtain the coefficients of the second fundamental form and the connection forms as

$$h^3_{11} = \kappa, \qquad h^3_{22} = -\frac{w''}{\kappa w}, \quad h^3_{12} = 0, \tag{47}$$

$$h^4_{11} = h^4_{12} = 0, \qquad h^4_{22} = -\frac{\rho}{\kappa w}, \tag{48}$$

$$\omega_{12}(e_1) = 0, \qquad \omega_{12}(e_2) = \frac{w'}{w}, \tag{49}$$

$$\omega_{34}(e_1) = \tau, \qquad \omega_{34}(e_2) = 0, \tag{50}$$

where τ is the torsion of the profile curve β. Hence, the mean curvature vector and the Gaussian curvature of M_2 are, respectively, given by

$$H = -\frac{1}{2}\Big[\Big(\frac{w''}{\kappa w} + \kappa\Big)e_3 + \frac{\rho}{\kappa w}e_4\Big] \quad \text{and} \quad K = -\frac{w''}{w}. \tag{51}$$

On the other hand, using the Codazzi equation (5) we get

$$e_1(h^3_{22}) = -\omega_{12}(e_2)\left(h^3_{11} + h^3_{22}\right) + \tau h^4_{22}, \tag{52}$$

$$e_1(h^4_{22}) = -\omega_{12}(e_2)h^4_{22} - \tau h^3_{22}. \tag{53}$$

Theorem 3.4. *Let M_2 be a flat timelike rotational surface of hyperbolic type in $\mathbb{E}^4_1$ defined by* (10)*. If the smooth timelike profile curve $\beta(s) = (x(s), y(s), 0, w(s))$ has the nonnull principal curvature vector $\beta''(s)$, then M_2 has pointwise 1-type Gauss map of the second kind if and only if $\beta(s)$ is given by*

$$x(s) = \frac{\sqrt{w_0^2 - 1}}{w_0(1 + q_0^2)}(w_0 s + w_1)(q_0 \sin\psi(s) + \cos\psi(s)), \tag{54}$$

$$y(s) = \frac{\sqrt{w_0^2 - 1}}{w_0(1 + q_0^2)}(w_0 s + w_1)(\sin\psi(s) - q_0 \cos\psi(s)), \tag{55}$$

$$w(s) = w_0 s + w_1,$$

$$\psi(s) = q_0 \ln(w_0 s + w_1) + \psi_0$$

and the Gauss map $\nu = e_3 \wedge e_4$ satisfies (1) *for the function*

$$f(s,t) = \frac{\kappa_0^2 + w_0^2 - 1}{(1 - w_0^2)(w_0 s + w_1)^2}$$

and for the constant vector

$$C = -w_0^2 e_3 \wedge e_4 + \frac{q_0 w_0^2 (w_0^2 - 1)}{\kappa_0} e_1 \wedge e_3$$

where $s > -w_1/w_0$, $|w_0| > 1$, $q_0 = \pm \frac{\kappa_0}{w_0\sqrt{w_0^2-1}}$ and $\kappa_0, w_0, w_1, \psi_0 \in \mathbb{R}$ with $\kappa_0 > 0$. Moreover, the profile curve β is a helix.

Proof. Let M_2 be a timelike rotational surface in $\mathbb{E}^4_1$ defined by (10). Suppose that the principal direction $\beta''(s)$ is nonnull. Then we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M_2 in $\mathbb{E}^4_1$ given by (44)–(46), and the coefficients of the second fundamental form are given by (47) and (48). Hence the Laplacian of the Gauss map $\nu = e_3 \wedge e_4$ from (13) is obtained as

$$\Delta\nu = \|h\|^2 \nu + (e_1(h^4_{22}) + \tau(h^3_{22} - h^3_{11}))e_1 \wedge e_3 + (e_1(h^3_{11} - h^3_{22}) + \tau h^4_{22})e_1 \wedge e_4.$$

If we use the Codazzi equations (52) and (53), then we get

$$\Delta\nu = \|h\|^2 \nu - (\tau h^3_{11} + \omega_{12}(e_2)h^4_{22})e_1 \wedge e_3 + (\kappa' + \omega_{12}(e_2)(h^3_{11} + h^3_{22}))e_1 \wedge e_4. \tag{56}$$

Suppose that M_2 is flat and has pointwise 1-type Gauss map of the second kind. Then, from the second of (51) we have $w'' = 0$, that is, $w(s) = w_0 s + w_1$ with $w_0, w_1 \in \mathbb{R}$, and hence $h^3_{22} = 0$ because of (47). Considering (56) and $h^3_{22} = 0$, we obtain

$$f(1 + C_{34}) = \|h\|^2 = (h^3_{11})^2 + (h^4_{22})^2, \tag{57}$$

$$fC_{13} = \tau h^3_{11} + \omega_{12}(e_2)h^4_{22}, \tag{58}$$

$$fC_{14} = -\kappa' - \omega_{12}(e_2)h^3_{11}, \tag{59}$$

$$C_{12} = C_{23} = C_{24} = 0. \tag{60}$$

For $i = 2$, from (15), (18) and (19) we have, respectively,

$$h^4_{22}C_{14} = 0, \tag{61}$$

$$\omega_{12}(e_2)C_{13} - h^4_{22}C_{34} = 0, \tag{62}$$

$$\omega_{12}(e_2)C_{14} = 0. \tag{63}$$

From (49), we have $\omega_{12}(e_2) = \dfrac{w_0}{w}$, and thus (63) implies that $w_0 C_{14} = 0$. Since β is parametrized by arc length parameter, that is, $x'^2 + y'^2 - w_0^2 = -1$, w_0 must be non-zero constant. Therefore, $C_{14} = 0$. So, we get $\kappa' + \dfrac{w_0}{w}\kappa = 0$ from (59) which yields $\kappa = \dfrac{\kappa_0}{w} = \dfrac{\kappa_0}{w_0 s + w_1}$ for some constant $\kappa_0 > 0$. Now, since $\beta'(s) = (x'(s), y'(s), 0, w_0)$ and $\langle \beta', \beta' \rangle = -1$ we get $x'^2 + y'^2 = -1 + w_0^2$. We put $\mu_0^2 = w_0^2 - 1 > 0$ where $|w_0| > 1$. Then $x'^2 + y'^2 = \mu_0^2$. Without loss of generality, we assume that $\mu_0 > 0$. When we write $x'(s) = \mu_0 \cos\psi(s)$ and $y'(s) = \mu_0 \sin\psi(s)$, we get $x''(s) = -\mu_0\psi'(s)\sin\psi(s)$ and $y''(s) = \mu_0\psi'(s)\cos\psi(s)$. Thus, $\kappa^2 = \langle \beta'', \beta'' \rangle = x''^2 + y''^2 = \mu_0^2\psi'^2$ from which we obtain $\psi(s) = q_0 \ln(w_0 s + w_1) + \psi_0$, where $q_0 = \pm\frac{\kappa_0}{w_0\sqrt{w_0^2-1}}$ and ψ_0 is an integration constant. Therefore, we obtain (54) and (55).

Also, we have $\rho = y'x'' - x'y'' = -\mu_0^2\psi' = -\mu_0^2\dfrac{q_0 w_0}{w}$. Hence, $h_{22}^4 = -\dfrac{\rho}{\kappa w} = \mu_0^2\dfrac{q_0 w_0}{\kappa_0 w}$ and equation (62) gives

$$\kappa_0 C_{13} - \mu_0^2 q_0 C_{34} = 0. \tag{64}$$

Now, by a calculation we have $\tau h_{11}^3 + \omega_{12}(e_2)h_{22}^4 = \dfrac{q_0 w_0^2(\kappa_0^2 + \mu_0^2)}{\kappa_0 w^2}$ and $||h||^2 = (h_{11}^3)^2 + (h_{22}^4)^2 = \dfrac{\kappa_0^2 + \mu_0^2}{w^2}$. From (57) and (58), we get $\kappa_0 C_{13} - q_0 w_0^2 C_{34} = q_0 w_0^2$ from which and (64) we obtain that $C_{13} = -\dfrac{(w_0^2-1)w_0^2 q_0}{\kappa_0}$ and $C_{34} = -w_0^2$. Hence, we get $C = -w_0^2 e_3 \wedge e_4 + \dfrac{w_0^2(w_0^2-1)q_0}{\kappa_0} e_1 \wedge e_3$ by (14). Also, from equation (57) we have $f(s,t) = \dfrac{w_0^2 + \kappa_0^2 - 1}{(1-w_0^2)(w_0 s + w_1)^2}$.

Considering the curve $\beta(s)$ lying in $\mathbb{E}_1^3$, we obtain $\tau = \dfrac{\langle \beta' \times \beta'', \beta''' \rangle}{\kappa^2} = \dfrac{q_0 w_0^2}{w}$ by a direct computation. Note that τ/κ is a constant, and thus the profile curve β is a helix.

The converse of the proof follows from a direct calculation. □

Note that if $\beta''(s) = 0$, that is, β is line, then it can be shown that M_2 lies in Euclidean space $\mathbb{E}^3$ or Minkowski space $\mathbb{E}_1^3$. Rotational surfaces in Euclidean space $\mathbb{E}^3$ or Minkowski space $\mathbb{E}_1^3$ with pointwise 1-type Gauss map were studied in [7, 13].

3.3. Rotational surfaces of parabolic type with pointwise 1-type Gauss Map

In this subsection, we prove that there exists no flat timelike rotational surfaces M_3 of parabolic type in $\mathbb{E}_1^4$ defined by (12) with pointwise 1-type Gauss map.

Let M_3 be a timelike rotational surface of parabolic type in $\mathbb{E}_1^4$ defined by (12). Suppose that the smooth timelike profile curve $\gamma : I \to \mathbb{E}_1^3$, $\gamma(s) = x(s)\eta_1 + z(s)\xi_3 + w(s)\xi_4$ is parametrized by the arc lenght, i.e., $x'^2(s) - 2z'(s)w'(s) = -1$, where I is an open interval in $\mathbb{R}$.

For the surface M_3, we consider the following orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ such that e_1, e_2 are tangent to M_3, and e_3, e_4 are normal to M_3:

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{\sqrt{2}w(s)}\frac{\partial}{\partial t}, \quad w(s) > 0, \tag{65}$$

$$e_3 = x'(s)\eta_1 + \sqrt{2}tw'(s)\eta_2 + \Big(-\frac{1}{w'(s)} + t^2 w'(s) + z'(s)\Big)\xi_3 + w'(s)\xi_4, \tag{66}$$

$$e_4 = \frac{x'(s)}{w'(s)}\xi_3 + \eta_1, \tag{67}$$

where $s \in I$, $\varepsilon_1 = \langle e_1, e_1 \rangle = -1$, $\varepsilon_2 = \langle e_2, e_2 \rangle = 1$, $\varepsilon_3 = \langle e_3, e_3 \rangle = 1$ and $\varepsilon_4 = \langle e_4, e_4 \rangle = 1$.

By a direct computation, we have the coefficients of the second fundamental form and the connection forms as follows

$$h^3_{11} = \frac{w''}{w'}, \qquad h^3_{22} = -\frac{w'}{w}, \quad h^3_{12} = h^3_{21} = 0, \tag{68}$$

$$h^4_{11} = \frac{x''w' - x'w''}{w'}, \qquad h^4_{22} = h^4_{12} = h^4_{21} = 0, \tag{69}$$

$$\omega_{12}(e_1) = 0, \qquad \omega_{12}(e_2) = \frac{w'}{w}, \tag{70}$$

$$\omega_{34}(e_1) = \frac{x''w' - x'w''}{w'}, \qquad \omega_{34}(e_2) = 0. \tag{71}$$

Then we obtain the mean curvature vector and Gauss curvature of M_3, respectively, as

$$H = -\frac{1}{2}\Big[\Big(\frac{w''}{w'} + \frac{w'}{w}\Big)e_3 + \Big(\frac{x''w' - x'w''}{w'}\Big)e_4\Big] \quad \text{and} \quad K = -\frac{w''}{w}. \tag{72}$$

On the other hand, from the Codazzi equation (5) we have

$$e_1(h^3_{22}) \;=\; -\omega_{12}(e_2)\left(h^3_{11} + h^3_{22}\right), \tag{73}$$

$$h^4_{11}\omega_{12}(e_2) \;+\; h^3_{22}\omega_{34}(e_1) = 0. \tag{74}$$

Note that $w'(s) \neq 0$ because the timelike profile curve γ is parametrized by arc lenght parameter s.

Theorem 3.5. *There exists no flat timelike rotational surface of the parabolic type defined by* (12) *in* $\mathbb{E}^4_1$ *with pointwise 1-type Gauss map.*

Proof. Let M_3 be a flat timelike rotational surface in $\mathbb{E}^4_1$ defined by (12). Then we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M_3 in $\mathbb{E}^4_1$ given by (65)–(67), and the components of the second fundamental form are given by (68) and (69). Hence the Laplacian of the Gauss map $\nu = e_3 \wedge e_4$ from (13) is obtained as

$$\Delta\nu = \|h\|^2\nu + ((h^3_{22} - h^3_{11})\omega_{34}(e_1) - e_1(h^4_{11}))e_1 \wedge e_3 + (e_1(h^3_{11} - h^3_{22}) - \omega_{34}(e_1)h^4_{11}))e_1 \wedge e_4.$$

If we use the Codazzi equation (73), then we get

$$\Delta\nu = \|h\|^2\nu - (e_1(h^4_{11}) - (h^3_{22} - h^3_{11})\omega_{34}(e_1))e_1 \wedge e_3 + (e_1(h^3_{11}) + \omega_{12}(e_2)(h^3_{11} + h^3_{22}) - h^4_{11}\omega_{34}(e_1))e_1 \wedge e_4. \tag{75}$$

According to the hypothesis, from the second of (51) we get $w'' = 0$, that is, $w(s) = w_0 s + w_1$ with $w_0, w_1 \in \mathbb{R}$ and $w_0 \neq 0$. Thus, from (68)–(71), we obtain

$$\begin{aligned} h^3_{11} &= 0, \quad h^3_{22} = -\frac{w_0}{w}, \quad h^3_{12} = h^3_{21} = 0, \\ h^4_{11} &= x'', \quad h^4_{22} = h^4_{12} = h^4_{21} = 0, \\ \omega_{12}(e_1) &= 0, \quad \omega_{12}(e_2) = \frac{w_0}{w}, \\ \omega_{34}(e_1) &= x'', \quad \omega_{34}(e_2) = 0. \end{aligned} \tag{76}$$

Considering (75) and (76), we have

$$f(1 + C_{34}) = \frac{w_0^2}{w^2} + x''^2, \tag{77}$$

$$fC_{13} = x''' + \frac{w_0}{w}x'', \tag{78}$$

$$fC_{14} = \frac{w_0^2}{w^2} + x''^2, \tag{79}$$

$$C_{12} = C_{23} = C_{24} = 0. \tag{80}$$

Since $w_0 \neq 0$ and C_{14} satisfies (79), C_{14} must be nonzero. Therefore, the Gauss map of M_3 is not pointwise 1-type Gauss map of the first kind.

Now, assume that Gauss map ν is of pointwise 1-type Gauss map of the second kind. For $i = 2$, from (15), (19) and (76) we have, respectively,

$$w_0 C_{13} = 0, \tag{81}$$

$$w_0(C_{14} - C_{34}) = 0 \tag{82}$$

because of $w_0 \neq 0$, $C_{13} = 0$ and $C_{14} = C_{34}$. However, by using the equations (77) and (79), we get $C_{14} - C_{34} = 1$ which is a contradiction to $C_{14} = C_{34}$. Consequently, M_3 has no pointwise 1-type Gauss map. □

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