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NABLA FRACTIONAL BOUNDARY VALUE PROBLEM WITH A NON-LOCAL BOUNDARY CONDITION

N. S. GOPAL[1], J. M. JONNALAGADDA[1*], §

Abstract. In this work, we deal with the following two-point boundary value problem for a finite fractional nabla difference equation with non-local boundary condition:

$$\begin{cases} -\big(\nabla^{\xi}_{\rho(e)}u\big)(z) = p(z,u(z)), & z \in \mathbb{N}^{f}_{e+2}, \\ u(e) = g(u), \quad u(f) = 0. \end{cases}$$

Here e, $f \in \mathbb{R}$, with $f - e \in \mathbb{N}_3$, $1 < \xi < 2$, $p : \mathbb{N}^{f}_{e+2} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, the functional $g \in C[\mathbb{N}^{f}_{e} \to \mathbb{R}]$ and $\nabla^{\xi}_{\rho(e)}$ denotes the ξ^{th}- order Riemann–Liouville backward (nabla) difference operator.

First, we derive the associated Green's function and some of its properties. Using the Guo–Krasnoselskii fixed point theorem on a suitable cone and under appropriate conditions on the non-linear part of the difference equation, we establish sufficient conditions for the existence of at least one positive solution to the boundary value problem. Next, we discuss the uniqueness of the solution to the considered problem. For this purpose, we use Brouwer and Banach fixed point theorem respectively. Finally, we provide an example to illustrate the applicability of established results.

Keywords: Nabla fractional difference, boundary value problem, positive solution, fixed point, existence.

AMS Subject Classification: 39A12.

1. Introduction

Over the last few decades, the theory of fractional calculus has been extensively developed due to its properties, generalising most results of differential calculus and its non-local nature of fractional derivatives. A robust theory of fractional differential equations for functions of a real variable has arisen from the contributions of various mathematicians over the course of three centuries. Its origins can be traced back to the letter written by

[1] Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad, 500078, Telangana, India.
e-mail: nsgopal94@gmail.com; ORCID: https://orcid.org/0000-0002-1166-3446.
e-mail: j.jaganmohan@hotmail.com; ORCID: https://orcid.org/0000-0002-1310-8323.
[*] Corresponding author.
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Leibniz on September 30, 1695. In the last three decades, fractional calculus has been successfully employed for mathematical modelling in health research, computational biology, finance, physics, and a variety of engineering subjects and also while doing so, various definitions of fractional derivatives were put forward, enabling its theory and applications in new directions, such a recent few can be found in [26, 27, 28, 29]. We refer to a few classic texts on fractional calculus by the following authors for more applications and historical literature Miller–Ross [24], Samko et al. [30], Podlubny [25] and Kilbas et al. [22].

On the other side of the coin, fractional nabla calculus is a branch of mathematics that deals with arbitrary order differences and sums in the backward sense. The theory of fractional nabla calculus is relatively young, with the most prominent works done in the past decade. The notion of nabla fractional difference and sum can be traced back to the work of Gray and Zhang[11] and Miller and Ross[24]. Following their work, the contributions of several mathematicians have made this theory a fruitful field of research in science and engineering. We refer here to a recent monograph by Goodrich and Peterson[7] and the references therein, which is an excellent source for all those who wish to work in this field.

The study of boundary value problems (BVPs) has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth in interest in the development of fractional nabla BVPs. Many authors have studied fractional nabla BVPs recently. To name a few, Ahrendt et al. [4], Goar [10], and Ikram [20] worked with self-adjoint Caputo nabla BVP. Brackins [5] studied a particular class of self-adjoint Riemann–Liouville nabla BVP and derived the Green's function associated with it along with a few of its properties. Gholami et al. [6] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [12, 13, 14, 15, 16, 17] analysed some qualitative properties of two-point non-linear Riemann–Liouville nabla BVPs associated with a variety of boundary conditions. Goodrich [8] has analysed fractional BVPs with a non-local condition in the delta case.

We consider the following boundary value problem with non-local condition

$$\begin{cases} -(\nabla^{\xi}_{\rho(e)}u)(z) = p(z, u(z)), \quad z \in \mathbb{N}^{f}_{e+2} \\ u(e) = g(u), \quad u(f) = 0, \end{cases} \tag{1}$$

where $e, f \in \mathbb{R}$, with $f - e \in \mathbb{N}_3, 1 < \xi < 2$, $p : \mathbb{N}^{f}_{e+2} \times \mathbb{R} \to \mathbb{R}$ and the functional $g \in C[\mathbb{N}^{f}_{e} \to \mathbb{R}]$ is continuous.

The present article is organised as follows: Section 2 contains a few preliminaries on discrete fractional nabla calculus. In Section 3, we construct Green's function and state a few of its properties. In Section 4, we establish the existence of the positive solution using Guo-Kranoselskii fixed point theorem on cones. In Section 5, we obtain sufficient conditions on the uniqueness of the solution for the proposed class of boundary value problem using Brouwer and Contraction mapping theorems, respectively. Finally, we conclude this article with an example.

2. Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. We will use the following notations, definitions, and known results of fractional nabla calculus [2, 7]. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1. *For $e \in \mathbb{R}$, the sets $\mathbb{N}_e$ and $\mathbb{N}_e^f$, where $f-e \in \mathbb{Z}^+$, are defined by*

$$\mathbb{N}_e = \{e, e+1, e+2, ...\}, \qquad \mathbb{N}_e^f = \{e, e+1, e+2, ..., f\}.$$

Definition 2.2. *We define the backward jump operator, $\rho : \mathbb{N}_{e+1} \longrightarrow \mathbb{N}_e$, by*

$$\rho(z) = z-1, \qquad z \in \mathbb{N}_{e+1}.$$

Let $v : \mathbb{N}_e \to \mathbb{R}$ and $M \in \mathbb{N}_1$. The first order nabla difference of v is defined by $\big(\nabla v\big)(z) = v(z) - v(z-1)$ for $z \in \mathbb{N}_{e+1}$, and the M^{th}-order nabla difference of v is defined recursively by $\big(\nabla^M v\big)(z) = \Big(\nabla\big(\nabla^{M-1} v\big)\Big)(z)$ for $z \in \mathbb{N}_{e+M}$.

Definition 2.3. *Let $z \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$ and $r \in \mathbb{R}$ such that $(z+r) \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$, the generalized rising function is defined by $z^{\overline{r}} = \frac{\Gamma(z+r)}{\Gamma(z)}$, $0^{\overline{r}} = 0$. Here $\Gamma(\cdot)$ denotes the Euler gamma function.*

Definition 2.4 (See [7, 17])**.** *Let $\kappa \in \mathbb{R} \setminus \{\dots,-2,-1\}$. The κ^{th}-order fractional nabla Taylor monomial is given by*

$$H_\kappa(z,e) = \frac{(z-e)^{\overline{\kappa}}}{\Gamma(\kappa+1)} = \frac{\Gamma(z-e+\kappa)}{\Gamma(z-e)\Gamma(\kappa+1)},$$

given the right-hand side exists.

We observe the following properties of the fractional nabla Taylor monomials,

(1) $H_\kappa(e,e) = 0$,
(2) $\nabla H_\kappa(z,e) = H_{\kappa-1}(z,e)$,
(3) $\displaystyle\sum_{s=e+1}^{z} H_\kappa(s,e) = H_{\kappa+1}(z,e)$,
(4) $\displaystyle\sum_{s=e+1}^{z} H_\kappa(z,\rho(s)) = H_{\kappa+1}(z,e)$,
(5) $H_{-k}(z,e) = 0, \ \forall\ k \in \mathbb{N}_1$,
(6) $H_{k+1}(z,\rho(e)) - H_k(z,\rho(e)) = H_{k+1}(z,e)$.

Lemma 2.1 (See [17, 20])**.** *Let $\kappa > -1$ and $s \in \mathbb{N}_e$. Then the following hold:*

(1) *If $z \in \mathbb{N}_{\rho(s)}$, then $H_\kappa(z,\rho(s)) \geq 0$, and if $z \in \mathbb{N}_s$, then $H_\kappa(z,\rho(s)) > 0$.*
(2) *If $z \in \mathbb{N}_s$ and $-1 < \kappa < 0$, then $H_\kappa(z,\rho(s))$ is an increasing function of s.*
(3) *If $z \in \mathbb{N}_{s+1}$ and $-1 < \kappa < 0$, then $H_\kappa(z,\rho(s))$ is a decreasing function of z.*
(4) *If $z \in \mathbb{N}_{\rho(s)}$ and $\kappa > 0$, then $H_\kappa(z,\rho(s))$ is a decreasing function of s.*
(5) *If $z \in \mathbb{N}_{\rho(s)}$ and $\kappa \geq 0$, then $H_\kappa(z,\rho(s))$ is a non-decreasing function of z.*
(6) *If $z \in \mathbb{N}_s$ and $\kappa > 0$, then $H_\kappa(z,\rho(s))$ is an increasing function of z.*
(7) *If $0 < v \leq \kappa$ then $H_v(t,a) \leq H_\kappa(z,e)$, for each fixed $z \in \mathbb{N}_e$.*
(8) $\frac{(e-s)^{\overline{\xi-1}}}{(f-s)^{\overline{\xi-1}}}$ *is a decreasing function of s for $s \in \mathbb{N}_0^{e-1}$, where $1 < \xi < 2$ and $f > e$.*

Proof. (8) It is enough to show that $\nabla_s \left(\frac{(e-s)^{\overline{\xi-1}}}{(f-s)^{\overline{\xi-1}}} \right) < 0$.

$$\nabla_s \left(\frac{(e-s)^{\overline{\xi-1}}}{(f-s)^{\overline{\xi-1}}} \right) = \frac{-(f-s)^{\overline{\xi-1}}(\xi-1)(e-\rho(s))^{\overline{\xi-2}} + (e-s)^{\overline{\xi-1}}(\xi-1)(f-\rho(s))^{\overline{\xi-2}}}{(f-s)^{\overline{\xi-1}}(f-\rho(s))^{\overline{\xi-1}}}$$

$$= \frac{(\xi-1)\left(-(f-s)(f-\rho(s))^{\overline{\xi-2}}(e-\rho(s))^{\overline{\xi-2}} + (e-s)(e-\rho(s))^{\overline{\xi-2}}(f-\rho(s))^{\overline{\xi-2}}\right)}{(f-s)^{\overline{\xi-1}}(f-\rho(s))^{\overline{\xi-1}}}$$

$$= \frac{(\xi-1)(f-\rho(s))^{\overline{\xi-2}}(e-\rho(s))^{\overline{\xi-2}}(-f+s+e-s)}{(f-s)^{\overline{\xi-1}}(f-\rho(s))^{\overline{\xi-1}}}.$$

Since $f > e$, we have $(-f+e) < 0$. Hence the result follows. The proof is complete. □

Definition 2.5 (See [7]). *Let $v : \mathbb{N}_{e+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of v is given by* $\left(\nabla_e^{-\nu} v\right)(z) = \sum_{s=e+1}^{z} H_{\nu-1}(z, \rho(s))u(s), \quad z \in \mathbb{N}_{e+1}$.

Definition 2.6 (See [7]). *Let $v : \mathbb{N}_{e+1} \to \mathbb{R}$, $\nu > 0$ and choose $M \in \mathbb{N}_1$, such that $M - 1 < \nu \le M$. The ν^{th}-order Riemann–Liouville nabla difference of v is given by*

$$\left(\nabla_e^{\nu} v\right)(z) = \left(\nabla^M \left(\nabla_e^{-(M-\nu)} v\right)\right)(z), \quad z \in \mathbb{N}_{e+M}.$$

Lemma 2.2 (See [2, 7]). *The following fractional nabla Taylor monomials are well defined.*

1. *Let $\nu > 0$ and $\kappa \in \mathbb{R}$. Then, $\nabla_e^{-\nu} H_\kappa(z, \rho(e)) = H_{\kappa+\nu}(z, \rho(e))$, $\quad z \in \mathbb{N}_e$.*
2. *Let ν, $\kappa \in \mathbb{R}$ and $M \in \mathbb{N}_1$ such that $M - 1 < \nu \le M$. Then,*

$$\nabla_e^{\nu} H_\kappa(z, \rho(e)) = H_{\kappa-\nu}(z, \rho(e)), \quad z \in \mathbb{N}_{e+M}.$$

We state here the nabla Laplace transform, and a few properties for the same, from [7].

Definition 2.7 (See [4, 7]). *Assume $p : \mathbb{N}_{e+1} \to \mathbb{R}$. Then, the nabla Laplace transform of p is defined by $\mathcal{L}_e[p(r)] = \sum_{k=1}^{\infty}(1-r)^{k-1}p(e+k)$, for those values of r, such that this infinite series converges.*

Theorem 2.1 (See [4, 7]). ***A****ssume $v > 0$ and the nabla Laplace transform of $p : \mathbb{N}_{e+1} \to \mathbb{R}$ converges for $|r-1| < s$ for some $s > 0$. Then, for $|r-1| < \min\{1, s\}$,*

$$\mathcal{L}_e\{\nabla_e^{-v} p\}(r) = \frac{1}{r^v}\mathcal{L}_e\{p\}(r).$$

Theorem 2.2 (See [7]). *Assume $p : \mathbb{N}_{e+1} \to \mathbb{R}$ has exponential order $s > 0$ and $0 < v < 1$. Then for $|r-1| < s$, $\mathcal{L}_e\{\nabla_e^{v} p\}(r) = r^v \mathcal{L}_e\{p\}(r)$.*

Lemma 2.3 (See [7]). *Given $p : \mathbb{N}_{e+1} \to \mathbb{R}$ and $q \in \mathbb{N}_1$, we have that*

$$\mathcal{L}_{e+q}\{p\}(s) = \left(\frac{1}{1-s}\right)^q \mathcal{L}_e\{p\}(s) - \sum_{k=1}^{q} \frac{f(e+k)}{(1-s)^{q-k+1}}.$$

Theorem 2.3 (See [7]). *Assume $v > 0$ and $M - 1 < v \le M$. Then, a general solution of $\nabla_e^v x(z) = 0$ is given by $x(z) = c_1 H_{v-1}(z, e) + c_2 H_{v-2}(z, e) + \cdots + c_M H_{v-M}(z, e)$, for $z \in \mathbb{N}_e$.*

3. Construction of Green's Function

In this section, we establish a formula for the Green's function of our boundary value problem (1) and also establish a few properties of the same, which will be used in the rest of the sections.

Theorem 3.1 (See [5, 12]). *The fractional nabla boundary value problem*

$$-(\nabla^{\xi}_{\rho(e)}u)(z) = h(z), \quad z \in \mathbb{N}^{f}_{e+2}, \qquad u(e) = u(f) = 0, \tag{2}$$

where $e, f \in \mathbb{R}$, with $f - e \in \mathbb{N}_3$, $1 < \xi < 2$ and $h : \mathbb{N}^{f}_{e+2} \to \mathbb{R}$, has the unique solution

$$u(z) = \sum_{s=e+2}^{f} G(z,s)h(s), \quad z \in \mathbb{N}^{f}_{e}, \tag{3}$$

where $G(z,s)$ is the Green's function, given by

$$G(z,s) = \begin{cases} G_1(z,s) = \frac{(z-e)^{\overline{\xi-1}}}{(f-e)^{\overline{\xi-1}}}\frac{(f-s+1)^{\overline{\xi-1}}}{\Gamma(\xi)}, & \text{for} \quad s > z, \\ G_2(z,s) = \frac{(z-e)^{\overline{\xi-1}}}{(f-e)^{\overline{\xi-1}}}\frac{(f-s+1)^{\overline{\xi-1}}}{\Gamma(\xi)} - \frac{(z-s+1)^{\overline{\xi-1}}}{\Gamma(\xi)}, & \text{for} \quad s \le z. \end{cases} \tag{4}$$

Lemma 3.1. *The solution of the homogeneous fractional nabla boundary value problem with non-local condition*

$$\begin{cases} -(\nabla^{\xi}_{\rho(e)}w)(z) = 0, \quad z \in \mathbb{N}^{f}_{e+2}, \\ w(e) = g(w), \quad w(f) = 0, \end{cases} \tag{5}$$

is given by

$$w(z) = g(w)\Big(\frac{f-z}{f-e}\Big)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}, \qquad z \in \mathbb{N}^{f}_{e}. \tag{6}$$

Proof. The general solution of the equation $-(\nabla^{\xi}_{\rho(e)}w)(z) = 0$, is given by

$$w(z) = c_1(z-e+1)^{\overline{\xi-1}} + c_2(z-e+1)^{\overline{\xi-2}}, \qquad z \in \mathbb{N}^{f}_{e}, \tag{7}$$

where c_1 and c_2 are arbitrary constants. Using $w(e) = g(w)$ and $w(f) = 0$, respectively in (7), we get that

$$\frac{g(w)}{\Gamma(\xi-1)} = c_1(\xi-1) + c_2,$$
$$0 = c_1(f-e+1)^{\overline{\xi-1}} + c_2(f-e+1)^{\overline{\xi-2}}.$$

Now, solving the above system of equations for c_1 and c_2, we get

$$c_1 = -\frac{g(w)(f-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)(f-e)^{\overline{\xi-1}}}, \quad c_2 = \frac{g(w)}{\Gamma(\xi-1)} + \frac{(\xi-1)g(w)(f-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)(f-e)^{\overline{\xi-1}}}.$$

Substituting c_1 and c_2 in (7) and using (6) of Definition 2.4, we get

$$\begin{aligned}
w(z) &= \left[-\frac{g(w)(f-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)(f-e)^{\overline{\xi-1}}}\right](z-e+1)^{\overline{\xi-1}} \\
&\quad + \left[\frac{g(w)}{\Gamma(\xi-1)} + \frac{(\xi-1)g(w)(f-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)(f-e)^{\overline{\xi-1}}}\right](z-e+1)^{\overline{\xi-2}} \\
&= g(w)\frac{(f-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)(f-e)^{\overline{\xi-1}}}\left[(z-e+1)^{\overline{\xi-2}}(\xi-1)-(z-e+1)^{\overline{\xi-1}}\right] \\
&\quad + g(w)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)} \\
&= \frac{g(w)}{\Gamma(\xi-1)}\left[(z-e+1)^{\overline{\xi-2}} - \frac{(f-e+1)^{\overline{\xi-2}}}{(f-e)^{\overline{\xi-1}}}(z-e)^{\overline{\xi-1}}\right] \\
&= \frac{g(w)}{\Gamma(\xi-1)}\left[\frac{(z-e+1)^{\overline{\xi-2}}}{(z-e)^{\overline{\xi-1}}} - \frac{(f-e+1)^{\overline{\xi-2}}}{(f-e)^{\overline{\xi-1}}}\right](z-e)^{\overline{\xi-1}} \\
&= \frac{g(w)}{\Gamma(\xi-1)}\left[\frac{1}{(z-e)} - \frac{1}{(f-e)}\right](z-e)^{\overline{\xi-1}} \\
&= \frac{g(w)}{\Gamma(\xi-1)}\left(\frac{f-z}{f-e}\right)\frac{(z-e)^{\overline{\xi-1}}}{(z-e)} \\
&= g(w)\left(\frac{f-z}{f-e}\right)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}.
\end{aligned}$$

The proof is complete. □

Lemma 3.2. *w satisfies the following property:*

$$\max_{z\in\mathbb{N}_e^f} w(z) \le g(w), \tag{8}$$

for all $w \in C[\mathbb{N}_e^f \to \mathbb{R}]$.

Proof. Consider

$$\max_{z\in\mathbb{N}_e^f} w(z) = \max_{z\in\mathbb{N}_e^f}\left(\frac{f-z}{f-e}\right)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}g(w)$$

From Lemma 2.1, it follows that the function $\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}$ is decreasing in terms of z. Thus, we have

$$\max_{z\in\mathbb{N}_e^f}\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)} = \frac{(e-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)} = 1,$$

and

$$\max_{z\in\mathbb{N}_e^f}\left(\frac{f-z}{f-e}\right) = 1.$$

Hence the result follows. The proof is complete. □

Theorem 3.2. *Let $p : \mathbb{N}_{e+2}^f \times \mathbb{R} \to \mathbb{R}$. The fractional nabla boundary value problem* (1) *has the unique solution*

$$u(z) = w(z) + \sum_{s=e+2}^{f} G(z,s)p(s,u(s)), \quad z \in \mathbb{N}_e^f,$$

where $G(z,s)$ and w is given by (4) and (6), respectively.

Theorem 3.3 (See [5, 12])**.** *The Green's function $G(z,s)$ defined in (4) satisfies the following properties:*

(1) $G(e,s) = G(f,s) = 0$, *for all* $s \in \mathbb{N}_{e+1}^f$.
(2) $G(z, e+1) = 0$, *for all* $z \in \mathbb{N}_e^f$.
(3) $G(z,s) > 0$, *for all* $(z,s) \in \mathbb{N}_{e+1}^{f-1} \times \mathbb{N}_{e+2}^f$.
(4) $\max_{z\in\mathbb{N}_{e+1}^{f-1}} G(z,s) = G(s-1,s)$, *for all* $s \in \mathbb{N}_{e+2}^f$.
(5) $\sum_{s=e+1}^{f} G(z,s) \leq \lambda$, *for all* $(z,s) \in \mathbb{N}_e^f \times \mathbb{N}_{e+1}^f$, *where*

$$\lambda = \Big(\frac{f-e-1}{\xi\Gamma(\xi+1)}\Big)\Big(\frac{(\xi-1)(f-e)+1}{\xi}\Big)^{\overline{\xi-1}}. \tag{9}$$

4. Positive Solutions

In this section, we prove the existence of at least one positive solution for the following standard non-linear fractional nabla boundary value problem with a non-local condition, using Guo–Krasnoselskii fixed point theorem [21, 23] on a suitable cone.

Definition 4.1. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{C}$ of $\mathcal{B}$ is said to be a cone provided,*
(i) $au + bv \in \mathcal{C}$, for all $u, v \in \mathcal{C}$ and all $a, b \geq 0$,
(ii) $u \in \mathcal{C}$ and $-u \in \mathcal{C}$ implies $u = 0$.

Definition 4.2. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Lemma 4.1 (See [1])**.** *[Guo—Krasnoselskii fixed point theorem] Let $\mathcal{B}$ be a Banach space and $\mathcal{C} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Further, assume that $T : \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow \mathcal{C}$ is a completely continuous operator. If, either*

(1) $\|Tv\| \leq \|v\|$ *for* $v \in \mathcal{C} \cap \partial\Omega_1$ *and* $\|Tv\| \geq \|v\|$ *for* $v \in \mathcal{C} \cap \partial\Omega_2$*; or*
(2) $\|Tv\| \geq \|v\|$ *for* $v \in \mathcal{C} \cap \partial\Omega_1$ *and* $\|Tv\| \leq \|v\|$ *for* $v \in \mathcal{C} \cap \partial\Omega_2$*;*

holds, then T has at least one fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We establish the following lemmas, which will be used later in this article.

Theorem 4.1. *There exits a number $\gamma \in (0,1)$, such that*

$$\min_{z\in\mathbb{N}_c^d} G(z,s) \geq \gamma \max_{t\in\mathbb{N}_e^f} G(z,s) = \gamma G(s-1,s), \tag{10}$$

where, $c, d \in \mathbb{N}_{e+1}^{f-1}$, $c = e + \left\lceil \frac{f-e+1}{4} \right\rceil$ and $d = e + 3\left\lfloor \frac{f-e+1}{4} \right\rfloor$.

Proof. Using the properties of the Green's function and Taylor monomial from Definition 2.4, Lemma 2.1 and Theorem 3.3 one has, for $s \in \mathbb{N}_{e+2}^f$,

$$\frac{G(z,s)}{G(s-1,s)} = \begin{cases} \frac{(z-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}}, & \text{for } s > z, \\ \frac{(z-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}} - \frac{(z-s+1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-s+1)^{\overline{\xi-1}}(s-e-1)^{\overline{\xi-1}}}, & \text{for } s \leq z. \end{cases}$$

Now, for $s > z$ and $c \le z \le d$, $G_1(z,s)$ is an increasing function with respect to z. Then,

$$\begin{aligned}\min_{z\in\mathbb{N}_c^d} G_1(z,s) =& G_1(c,s)\\ =& \frac{(c-e)^{\overline{\xi-1}}(f-s+1)^{\overline{\xi-1}}}{(f-e)^{\overline{\xi-1}}\Gamma(\xi)}.\end{aligned}$$

For $z \ge s$ and $c \le z \le d$, $G_2(z,s)$ is a decreasing function with respect to z. Then,

$$\begin{aligned}\min_{z\in\mathbb{N}_c^d} G_2(z,s) =& G_2(d,s),\\ =& \frac{(d-e)^{\overline{\xi-1}}(f-s+1)^{\overline{\xi-1}}}{(f-e)^{\overline{\xi-1}}\Gamma(\xi)} - \frac{(d-s+1)^{\overline{\xi-1}}}{\Gamma(\xi)}.\end{aligned}$$

Thus,

$$\begin{aligned}\min_{z\in\mathbb{N}_c^d} G(z,s) &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{e+2}^{c},\\ \min\left(G_2(d,s), G_1(c,s)\right), & \text{for } s \in \mathbb{N}_{c+1}^{d-1},\\ G_1(c,s), & \text{for } s \in \mathbb{N}_{d}^{f},\end{cases}\\ &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{e+2}^{r},\\ G_1(c,s), & \text{for } s \in \mathbb{N}_{r}^{f},\end{cases}\end{aligned}$$

where $c < r < d$. Consider

$$\frac{\min_{z\in\mathbb{N}_c^d} G(z,s)}{G(s-1,s)} = \begin{cases} \frac{(d-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}} - \frac{(d-s+1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-s+1)^{\overline{\xi-1}}(s-e-1)^{\overline{\xi-1}}}, & \text{for } s \in \mathbb{N}_{e+2}^{r},\\ \frac{(c-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}}, & \text{for } s \in \mathbb{N}_{r}^{f}.\end{cases}$$

Thus,

$$\min_{z\in\mathbb{N}_c^d} G(z,s) \ge \gamma(s) \max_{z\in\mathbb{N}_e^f} G(z,s), \tag{11}$$

where

$$\gamma(s) = \min\left[\frac{(c-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}}, \frac{(d-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}} - \frac{(d-s+1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-s+1)^{\overline{\xi-1}}(s-e-1)^{\overline{\xi-1}}}\right].$$

Let for $s \in \mathbb{N}_r^f$, denote

$$\begin{aligned}\gamma_1(s) =& \frac{(c-e)^{\overline{\xi-1}}}{(s-e-1)^{\overline{\xi-1}}}\\ \ge& \frac{(c-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}}.\end{aligned}$$

Similarly for $s \in \mathbb{N}_{e+2}^r$, we take

$$\gamma_2(s) = \frac{1}{(s-e-1)^{\overline{\xi-1}}}\left[(d-e)^{\overline{\xi-1}} - \frac{(d-s+1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-s+1)^{\overline{\xi-1}}}\right].$$

By Lemma 2.1, we see that $\frac{(d-s+1)^{\overline{\xi-1}}}{(f-s+1)^{\overline{\xi-1}}}$ is a decreasing function for $s \in \mathbb{N}_{e+2}^{r}$,

$$\begin{aligned}\gamma_2(s) \geq & \frac{1}{(s-e-1)^{\overline{\xi-1}}}\left[(d-e)^{\overline{\xi-1}} - \frac{(d-e-1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}}\right] \\ > & \frac{1}{(d-e)^{\overline{\xi-1}}}\left[(d-e)^{\overline{\xi-1}} - \frac{(d-e-1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}}\right].\end{aligned}$$

Thus,

$$\min_{z\in\mathbb{N}_c^d} G(z,s) \geq \gamma \max_{\in\mathbb{N}_e^f} G(z,s), \tag{12}$$

where

$$\gamma = \min\left[\frac{(c-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}}, 1 - \frac{(d-e-1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}(d-e)^{\overline{\xi-1}}}\right].$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\gamma(s) > 0$ for all $s \in \mathbb{N}_{e+2}^{f}$, implying $\gamma > 0$. It would be sufficient to prove that one of the terms $\frac{(c-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}}$, $1 - \frac{(d-e-1)^{\overline{\xi-1}}(f-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}(d-e)^{\overline{\xi-1}}}$ is less then 1. It follows from Lemma 2.1 that

$$\frac{(c-e)^{\overline{\xi-1}}}{(f-e-1)^{\overline{\xi-1}}} < 1.$$

Therefore we conclude that $\gamma \in (0,1)$. The proof is complete. □

Lemma 4.2. *If g is non-negative, then there exists a constant $\overline{\gamma} \in (0,1)$, such that*

$$\begin{aligned}&\min_{z\in\mathbb{N}_c^d} \sum_{s=e+2}^{f} G(z,s)p(s,u(s)) + \min_{z\in\mathbb{N}_c^d}\left[g(u)\Big(\frac{f-z}{f-e}\Big)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}\right] \\ \geq\ & \overline{\gamma}\max_{z\in\mathbb{N}_e^f} \sum_{s=e+2}^{f} G(z,s)p(s,u(s)) + \overline{\gamma}\max_{z\in\mathbb{N}_e^f}\left[g(u)\Big(\frac{f-z}{f-e}\Big)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}\right].\end{aligned}$$

Proof. From Theorem 4.1, we observe that there exits $\gamma \in (0,1)$, such that

$$\min_{z\in\mathbb{N}_c^d} G(z,s) \geq \gamma \max_{z\in\mathbb{N}_e^f} G(z,s),$$

thus, we have

$$\begin{aligned}\min_{z\in\mathbb{N}_c^d} \sum_{s=e+2}^{f} G(z,s)p(s,u(s)) \geq & \sum_{s=e+2}^{f} \min_{z\in\mathbb{N}_c^d} G(z,s)p(s,u(s)) \\ \geq & \gamma \sum_{s=e+2}^{f} \max_{z\in\mathbb{N}_e^f} G(z,s)p(s,u(s)) \\ \geq & \gamma \max_{z\in\mathbb{N}_e^f} \sum_{s=e+2}^{f} G(z,s)p(s,u(s)).\end{aligned}$$

Consider w,

$$w(z) = g(u)\Big(\frac{f-z}{f-e}\Big)\frac{(z-e+1)^{\overline{\xi-2}}}{\Gamma(\xi-1)}, \quad z \in \mathbb{N}_{e+2}^{f}.$$

From Lemma 3.2, we have
$$w(z) \leq g(u).$$
We deduce the existence of $M > 0$, such that
$$\min_{z \in \mathbb{N}_c^d} H_{\xi-2}(z, \rho(e)) \Big(\frac{f-z}{f-e} \Big) = M.$$
Let $\gamma_0 = M$,
$$\min_{z \in \mathbb{N}_c^d} H_{\xi-2}(z, \rho(e)) \Big(\frac{f-z}{f-e} \Big) g(u) = \gamma_0 \max_{z \in \mathbb{N}_e^f} H_{\xi-2}(z, \rho(e)) \Big(\frac{f-z}{f-e} \Big) g(u),$$
here $\max_{z \in \mathbb{N}_e^f} H_{\xi-2}(z, \rho(e)) \Big(\frac{f-z}{f-e} \Big) = 1$. Let $\overline{\gamma} = min[\gamma, \gamma_0] \Longrightarrow \overline{\gamma} \in (0,1)$. Thus
$$\min_{z \in \mathbb{N}_c^d} \sum_{s=e+2}^{f} G(z,s) p(s, u(s)) + \min_{z \in \mathbb{N}_c^d} w(z) \geq \overline{\gamma} \max_{z \in \mathbb{N}_e^f} \sum_{s=e+2}^{f} G(z,s) p(s, u(s)) + \overline{\gamma} \max_{z \in \mathbb{N}_e^f} w(z).$$
The proof is complete. □

We observe by Theorem 3.2 that, u is a solution of (1) if and only if u is a solution of the summation equation
$$u(z) = w(z) + \sum_{s=e+2}^{f} G(z,s) p(s, u(s)), \quad z \in \mathbb{N}_e^f. \tag{13}$$
Let us define the operator $T : \mathbb{R}^{f-e+1} \to \mathbb{R}^{f-e+1}$ by
$$Tu(z) = w(z) + \sum_{s=e+2}^{f} G(z,s) p(s, u(s)). \tag{14}$$

We also observe from (13) and (14), that u is a fixed point of T, if and only if u is a solution of (1). We use the fact that $\mathcal{B} = \mathbb{R}^{f-e+1}$ is a Banach space equipped with the maximum norm.
We define the cone $\mathcal{C}$, by
$$\mathcal{C} = \{u \in \mathcal{B} : \ u(z) \geq 0 \text{ and } \min_{z \in \mathbb{N}_c^d} u(z) \geq \overline{\gamma} \, \|u(z)\|\}. \tag{15}$$
Note that, T is a summation operator on a discrete finite set. Hence, operator T is trivially completely continuous. We state here the following hypotheses, which will be used later. Take
$$\eta = \frac{1}{\sum_{s=e+2}^{f} G(s-1, s)}. \tag{16}$$

- (H1) $p(z,u) \geq 0$, $(z,u) \in \mathbb{N}_e^f \times [0, \infty)$ and $g(u) \geq 0$, $\forall u \in C[\mathbb{N}_{e+2}^f \to \mathbb{R}]$,
- (H2) There exists a number $r_1 > 0$ such that $p(z,u) \leq \frac{r_1 \eta}{2}$, whenever $0 \leq u \leq r_1$,
- (H3) There exists a number $r_2 > 0$ such that $p(z,u) \geq \frac{r_2 \eta}{2\overline{\gamma}}$, whenever $\overline{\gamma} r_2 \leq u \leq r_2$,
- (H4) There exists a number $r_2 > 0$ such that $p(z,u) \leq \frac{r_2 \eta}{2}$, whenever $\overline{\gamma} r_2 \leq u \leq r_2$,
- (H5) There exists a number $r_1 > 0$ such that $p(z,u) \geq \frac{r_1 \eta}{2\overline{\gamma}}$, whenever $0 \leq u \leq r_1$,
- (G1) There exists a number $r_1 > 0$ such that $g(u) \leq \frac{r_1}{2}$, whenever $0 \leq u \leq r_1$,

(G2) There exists a number $r_2 > 0$ such that $g(u) \geq \dfrac{r_2}{2\overline{\gamma}}$, whenever $\overline{\gamma} r_2 \leq u \leq r_2$,

(G3) There exists a number $r_2 > 0$ such that $g(u) \leq \dfrac{r_2}{2}$, whenever $\overline{\gamma} r_2 \leq u \leq r_2$,

(G4) There exists a number $r_1 > 0$ such that $g(u) \geq \dfrac{r_1}{2\overline{\gamma}}$, whenever $0 \leq u \leq r_1$.

Lemma 4.3. *Assume* $(H1)$ *holds. Then,* $T : \mathcal{C} \to \mathcal{C}$.

Proof. Let T be the operator as defined in (14), then from Lemma 4.2, we have

$$\begin{aligned}
\min_{z\in\mathbb{N}_c^d}(Tu)(z) \geq& \min_{z\in\mathbb{N}_c^d}\sum_{s=e+2}^{f} G(z,s)p(s,u(s)) + \min_{z\in\mathbb{N}_c^d} w(z)\\
\geq& \overline{\gamma}\max_{z\in\mathbb{N}_e^f}\sum_{s=e+2}^{f} G(z,s)p(s,u(s)) + \overline{\gamma}\max_{z\in\mathbb{N}_e^f} w(z)\\
\geq& \overline{\gamma}\|Tu\|.
\end{aligned}$$

It is obvious that $Tu(z) \geq 0$, whenever $u \in \mathcal{C}$, thus $T : \mathcal{C} \to \mathcal{C}$. The proof is complete. $\square$

Theorem 4.2. *Assume* $p(z,u)$ *and* $g(u)$ *satisfy conditions* $\{$*(H1),(H2),(H3)*$\}$ *and* $\{$*(H1),(G1), (G2)*$\}$ *respectively. Then, the boundary value problem* (1) *has at least one positive solution.*

Proof. We know that, $T : \mathcal{C} \to \mathcal{C}$ is completely continuous. Define the set $\Omega_1 = \{u \in \mathcal{C} : \|u\| < r_1\}$ where $0 < r_1 < r_2$. Clearly, $\Omega_1 \subset \mathcal{B}$ is an open set with $0 \in \Omega_1$. Since $\|u\| = r_1$ for $u \in \partial\Omega_1$. Then, we have for $u \in \mathcal{C} \cap \partial\Omega_1$, condition (H2), (G1) holds.

$$\|Tu\| \leq \max_{z\in\mathbb{N}_e^f}\left|\sum_{s=e+2}^{f} G(z,s)p(s,u(s))\right| + \max_{z\in\mathbb{N}_e^f}|w(z)|,$$

using $(H2)$ and $(G1)$ we have

$$\begin{aligned}
\|Tu\| \leq& \sum_{s=e+2}^{f}\max_{z\in\mathbb{N}_e^f}[G(z,s)]p(s,u(s)) + g(u)\\
\leq& \frac{r_1\eta}{2}\sum_{s=e+2}^{f} G(s-1,s) + \frac{r_1}{2}\\
=& \frac{r_1}{2} + \frac{r_1}{2} = r_1.
\end{aligned}$$

Thus, we have $\|Tu\| \leq \|u\|$, for $u \in \mathcal{C} \cap \partial\Omega_1$. Similarly we define set $\Omega_2 = \{u \in \mathcal{C} : \|u\| < r_2\}$. Clearly, $\Omega_1 \in \beta$ is an open set and $\overline{\Omega}_1 \subseteq \Omega_2$. Since $\|u\| = r_2$ for $u \in \partial\Omega_2$, conditions

(H_3) and (G_2), holds for all $u \in \partial\Omega_2$, by using Lemma 3.2 and Lemma 4.2, we have

$$\begin{aligned}
\|Tu\| &\geq \min_{z\in\mathbb{N}_c^d} Tu(z)\\
&\geq \min_{z\in\mathbb{N}_c^d} \sum_{s=e+2}^{f} G(z,s)p(s,u(s)) + \min_{z\in\mathbb{N}_c^d} w(z)\\
&\geq \sum_{s=e+2}^{f} \min_{z\in\mathbb{N}_c^d}[G(z,s)]p(s,u(s)) + \min_{z\in\mathbb{N}_c^d} w(z)\\
&\geq \overline{\gamma} \sum_{s=e+2}^{f} \max_{z\in\mathbb{N}_e^f}[G(z,s)]p(s,u(s)) + \overline{\gamma}\max_{z\in\mathbb{N}_e^f} w(z)\\
&\geq \overline{\gamma}\frac{r_2\eta}{2\overline{\gamma}} \sum_{s=e+2}^{f} G(s-1,s) + \overline{\gamma}g(u)\\
&\geq \frac{r_2}{2} + \frac{r_2}{2} = r_2 = \|u\|.
\end{aligned}$$

Thus, we have $\|Tu\| \geq \|u\|$, for $u \in \mathcal{C} \cap \partial\Omega_2$. By part (1) of lemma 4.1, we conclude that, the operator T has at least one fixed point u_0 in $\mathcal{C} \cap (\Omega_2 \setminus \Omega_1)$, satisfying $r_1 < \|u_0\| < r_2$. The proof is complete. □

Theorem 4.3. *Assume $p(z,u)$ and $g(u)$ satisfy conditions {(H1),(H4),(H5)} and {(H1), (G3),(G4)} respectively. Then, the boundary value problem* (1) *has at least one positive solution.*

Proof. We know that, $T : \mathcal{C} \to \mathcal{C}$ is completely continuous. Define the set $\Omega_2 = \{u \in \mathcal{C} : \|u\| < r_2\}$ where $0 < r_1 < r_2$. Clearly, $\Omega_2 \subset \mathcal{B}$ is an open set and $\overline{\Omega}_1 \subseteq \Omega_2$. Since $\|u\| = r_2$ for $u \in \partial\Omega_2$. Then we have for $u \in \mathcal{C} \cap \partial\Omega_2$, condition (H4), (G3) holds.

$$\|Tu\| \leq \max_{z\in\mathbb{N}_e^f} \left| \sum_{s=e+2}^{f} G(z,s)p(s,u(s)) \right| + \max_{z\in\mathbb{N}_e^f} |w(z)|,$$

using $(H4)$ and $(G3)$, we have

$$\begin{aligned}
\|Tu\| &\leq \frac{r_2\eta}{2} \sum_{s=e+2}^{f} \max_{z\in\mathbb{N}_e^f}[G(z,s)] + \max_{z\in\mathbb{N}_e^f} w(z)\\
&\leq \frac{r_2\eta}{2} \sum_{s=e+2}^{f} G(s-1,s) + g(u)\\
&\leq \frac{r_2}{2} + \frac{r_2}{2} = r_2.
\end{aligned}$$

Thus, we have $\|Tu\| \leq \|u\|$, for $u \in \mathcal{C} \cap \partial\Omega_2$. Similarly we define set $\Omega_1 = \{u \in \mathcal{C} : \|u\| < r_1\}$. Clearly, $\Omega_1 \in \beta$ is an open set and $0 \in \Omega_1$. Since $\|u\| = r_1$ for $u \in \partial\Omega_1$ and conditions

(H5) and (G4), holds for all $u \in \partial\Omega_1$, by using Lemma 3.2, we have

$$\begin{aligned}
\|Tu\| &\geq \min_{z\in\mathbb{N}_c^d} Tu(z)\\
&\geq \min_{z\in\mathbb{N}_c^d} \sum_{s=e+2}^{f} G(z,s)p(s,u(s)) + \min_{z\in\mathbb{N}_c^d} w(z)\\
&\geq \sum_{s=e+2}^{f} \min_{z\in\mathbb{N}_c^d}[G(z,s)]p(s,u(s)) + \min_{z\in\mathbb{N}_c^d} w(z)\\
&\geq \overline{\gamma} \sum_{s=e+2}^{f} \max_{z\in\mathbb{N}_e^f}[G(z,s)]p(s,u(s)) + \overline{\gamma} \max_{z\in\mathbb{N}_e^f} w(z)\\
&\geq \overline{\gamma}\frac{r_1\eta}{2\overline{\gamma}} \sum_{s=e+2}^{f} G(s-1,s) + \overline{\gamma} g(u)\\
&\geq \frac{r_1}{2} + \frac{r_1}{2} = r_1 = \|u\|.
\end{aligned}$$

Thus, we have $\|Tu\| \geq \|u\|$, for $u \in \mathcal{C} \cap \partial\Omega_1$. By part (2) of lemma 4.1, we conclude that, the operator T has at least one fixed point u_0 in $\mathcal{C} \cap (\Omega_2 \setminus \Omega_1)$, satisfying $r_1 < \|u_0\| < r_2$. The proof is complete. □

5. Uniqueness of Solutions

In this section, we present uniqueness results of (1) using Brouwer fixed point theorem and contraction mapping theorem, respectively, and also we construct an example to illustrate the same results.

Theorem 5.1 (See [1])**.** *[Brouwer fixed point theorem]. Let $\mathcal{C}_0$ be a nonempty compact convex subset of $\mathbb{R}^n$ and T be a continuous mapping of $\mathcal{C}_0$ into itself. Then, T has a fixed point in $\mathcal{C}_0$.*

Theorem 5.2. *Assume $p(z,u)$ and $g(u)$ is continuous with respect to $'u'$, for each $z \in \mathbb{N}_e^f$. Assume there exist a positive constant L, such that*

$$\max_{-L\leq\|u\|\leq L}\{g(u)\} \leq \frac{L}{\lambda+1} \ \text{and} \ \max_{(z,u)\in\mathbb{N}_e^f\times[-L,L]}\{p(z,u)\} \leq \frac{L}{\lambda+1}. \tag{17}$$

Then, the boundary value problem (1) *has a solution.*

Proof. Consider $\mathcal{C}_0 = \{u \in \mathbb{N}_e^f \to \mathbb{R}, \|u\| \leq L\}$. Clearly, $\mathcal{C}_0$ is a non-empty compact convex subset of $\mathbb{R}^{f-e+1}$. Let T be a operator as defined in (14). It is clear that T is a continuous operator. Therefore the main objective is to show that $T : \mathcal{C}_0 \to \mathcal{C}_0$. Then, Theorem 5.1 can be invoked.

Let $\Omega_0 = \frac{L}{\lambda+1}$, By using (17) and Lemma 3.2

$$\begin{aligned}
\|Tu\| &\leq \max_{z\in\mathbb{N}_e^f}|w(z)| + \max_{z\in\mathbb{N}_e^f}\sum_{s=e+2}^{f} G(z,s)p(s,u(s)) \\
&\leq \Big(\frac{f-z}{f-e}\Big)H_{\xi-2}(z,\rho(e))g(u) + \Big(\frac{L}{\lambda+1}\Big)\max_{z\in\mathbb{N}_e^f}\sum_{s=e+2}^{f} G(z,s) \\
&\leq \Big(\frac{L}{\lambda+1}\Big) + \Big(\frac{L}{\lambda+1}\Big)\lambda \\
&= \Omega_0(1+\lambda) = L.
\end{aligned}$$

Thus, $\|Tu\| \leq L$ and $T : \mathcal{C}_0 \to \mathcal{C}_0$. It follows at once by Brouwer fixed point theorem, that there exist a fixed point of T say $u_0 \in \mathcal{C}$, such that $||u_0|| \leq L$. The proof is complete. □

Theorem 5.3 (See [1])**.** *[Contraction Mapping Theorem]. Let S be a closed subset of $\mathbb{R}^n$. Assume $T : S \to S$ is a contraction mapping, i.e there exists a number $'\kappa'$, $0 \leq \kappa \leq 1$, such that*
$\|Tu - Tv\| \leq \kappa\,\|u - v\|$, for all $u, v \in S$. Then, T has a unique fixed point $u_0 \in S$.

Theorem 5.4. *Assume that $p(z,u)$ and $g(u)$ are Lipschitz with respect to $'u'$, i.e there exists $\xi_1, \beta > 0$ such that $|p(z,u_1)-p(z,u_2)| \leq \xi_1\,\|u_1 - u_2\|$ and $|g(u_1)-g(u_2)| \leq \beta\,\|u_1 - u_2\|$ whenever $u_1, u_2 \in C[\mathbb{N}_e^f \to \mathbb{R}]$. Then, the boundary value problem (1), has a unique solution, provided $\xi_1\lambda + \beta < 1$ holds.*

Proof.

$$\begin{aligned}
\|Tu_1 - Tu_2\| &= \max_{t\in\mathbb{N}_e^f}|(Tu_1)(z) - (Tu_2)(z)| \\
&\leq \max_{z\in\mathbb{N}_e^f}\left|\sum_{s=e+2}^{f} G(z,s)p(s,u_1(s))\right| - \max_{z\in\mathbb{N}_e^f}\left|\sum_{s=e+2}^{f} G(z,s)p(s,u_2(s))\right| \\
&\quad + \max_{z\in\mathbb{N}_e^f}\left|\Big(\frac{f-z}{f-e}\Big)H_{\xi-2}(z,\rho(e))g(u_1)\right| - \max_{z\in\mathbb{N}_e^f}\left|\Big(\frac{f-z}{f-e}\Big)H_{\xi-2}(z,\rho(e))g(u_2)\right| \\
&\leq \max_{z\in\mathbb{N}_e^f}\sum_{s=e+2}^{f} G(z,s)|p(z,u_1) - p(z,u_2)| \\
&\quad + \max_{z\in\mathbb{N}_e^f}\left|\Big(\frac{f-z}{f-e}\Big)H_{\xi-2}(z,\rho(e))\right|\,|g(u_1) - g(u_2)| \\
&\leq \xi_1\,\|u_1 - u_2\|\max_{z\in\mathbb{N}_e^f}\sum_{s=e+2}^{f} G(z,s) + |g(u_1) - g(u_2)| \\
&\leq \xi_1\,\|u_1 - u_2\|\,\lambda + \beta\,\|u_1 - u_2\| \\
&= (\xi_1\lambda + \beta)\,\|u_1 - u_2\|\,.
\end{aligned}$$

Thus, T is a contraction on $\mathbb{R}^{f-e+1}$. Hence, by Theorem 5.3, the boundary value problem (1) has a unique solution. The proof is complete. □

Example 5.1. *Suppose, $\xi = 1.1,\ e = 0,\ f = 10$, $p(z,u) = \frac{sin(u)}{15+z}$ and $g(u) = \frac{u(z)}{20}$, $p(z,u)$ and $g(u)$ are Lipschitz with respect to u with Lipschitz constant ξ_1 and β respectively. Here*

$\xi_1 = \frac{1}{15}$ *and* $\beta = \frac{1}{20}$. *We have*

$$\begin{cases} -(\nabla^{1.1}_{\rho(0)} u)(z) = \frac{sin(u)}{15+z}, \\ u(0) = \frac{u(t)}{20}, \quad u(10) = 0. \end{cases} \tag{18}$$

Here

$$\lambda = \Big(\frac{(f-e)(\xi-1)+1}{\xi}\Big)^{\overline{\xi-1}} \Big(\frac{(f-e-1)}{\xi\Gamma(\xi+1)}\Big) = 8.05$$

$$(\xi_1\lambda + \beta) = 0.586 < 1.$$

Thus, by Theorem 5.4 the boundary value problem (18) has a unique solution.

6. Conclusions

In this work, we considered the two-point boundary value problem (1), we constructed the Green's function and established sufficient conditions on the existence of at least one positive solution to the considered problem using Guo–Krasnoselskii fixed point theorem. We also discussed the uniqueness of solution of (1) using Banach fixed point theorem and Contraction mapping theorem. The results obtained also hint at a possible direction in the field of fractional derivatives with q-calculus [18, 19]. Finally, we provided an example to illustrate the applicability of established results.

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Nandhihalli Srinivas Gopal obtained his Masters in Mathematics in 2017 from Bangalore University and cleared the CSIR-JRF exam in 2018. He is currently working in the Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad, India, as a senior research fellow (SRF) under the supervision of Dr. Jagan Mohan Jonnalagadda.

Jagan Mohan Jonnalagadda is currently working as an Associate Professor of Mathematics at Birla Institute of Technology and Science Pilani, Hyderabad, India. His research areas of interest are differential equations, difference equations, fractional differential equations, fractional difference equations and nonlinear analysis