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ON THE NORMS OF R-CIRCULANT MATRICES INVOLVING BALANCING AND LUCAS-BALANCING NUMBERS

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Abstract. In this article, we investigate the r-circulant matrices $B[r]$ and $C[r]$ involving the balancing and Lucas-balancing numbers respectively with arithmetic indices. For matrices $B[r]$ and $C[r]$, we establish the direct formula for the eigenvalues, the determinant, the Euclidean norm and the bound for the spectral norm. Furthermore, we extend the concept to right circulant matrices and skew-right circulant matrices and, investigate all the above results including the sum identities and divisibility.

Keywords: Balancing Number, Circulant Matrix, Determinant, Eigenvalues, Euclidean Norms, Spectral Norm.

AMS Subject Classification: 11B37, 65Q30, 11B39.

1. Introduction

Special number sequences like Fibonacci, Lucas, Horadam, Pell, Jacobsthal, Mersenne, etc., are widely studied topics in number theory. Especially generalizations of a number sequence, establishing new identities and their application in other branches have become very popular among the researchers (for example, see [1, 5, 11, 33, 20, 17]). Interestingly, Özkan, et al. [22] studied the relationship of new families for k-Lucas numbers with classical Fibonacci and Lucas numbers, later in [23] they studied new families of the Gauss k-Jacobsthal and Gauss k-Jacobsthal-Lucas numbers. In [26], Prasad et. al. proposed a generalized recursive matrix made with k-step Fibonacci numbers and shown their application in cryptography, later in [27], they extended the concept to generalized Lucas matrices. In [28], Ray et. al. studied the image scrambling with the balancing numbers and balancing transformations.

Construction of special matrices with a number sequence, investigation of its algebraic properties and their application in cryptography, image processing, signal processing, etc., are very interesting research subjects in matrix analysis, some work in this direction can

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be seen in [19, 24, 13]. In recent years, several articles on the construction of Toeplitz matrices, Hankel matrices and r-circulant matrices involving a number sequence appeared. The study shows the investigation of properties like some norms, the closed form formula for eigenvalues and the determinants involving the number sequences (see [7, 8, 9, 14, 15]). In the study of the construction of these matrices, a traditional way is to pick up a number sequence and make its terms as the entries of matrices and then investigate further properties.

Recently, A.C.F. Bueno studied the right circulant matrices with Pell and Pell-Lucas numbers [8] and Fibonacci numbers [6], after that they investigated r-circulant matrices with Fibonacci & Lucas numbers in [7] and Horadam numbers in [9] having arithmetic indices. Some works in this direction and their extension are due to Akbulak, et.al.[2], Gökbaş, et.al.[14, 15], Shen[30] and Solak[31].

These special matrices are widely used in solving different types of differential equations [32, 12], image and signal processing [3, 10] and vibration analysis[21], etc.

Two of such fascinating number sequences are the balancing numbers and Lucas-balancing numbers. The concept of balancing numbers and balancers was originally introduced in 1999 by Behera et al.[4].

A natural number n is said to be balancing number [4] with balancer r if it satisfy the Diophantine equation

$$1 + 2 + 3 + ... + (n-1) = (n+1) + (n+2) + ... + (n+r).$$

The Balancing number B_n and Lucas-balancing number C_n are defined by the recurrence relation

$$B_{n+2} = 6B_{n+1} - B_n, n \geq 0 \ with \ B_0 = 0, B_1 = 1, \tag{1}$$

$$C_{n+2} = 6C_{n+1} - C_n, n \geq 0 \ with \ C_0 = 1, B_1 = 3. \tag{2}$$

The first few terms of balancing and Lucas-balancing numbers are

n	**0**	**1**	2	3	4	5	6	7	8	...
B_n	**0**	**1**	6	35	204	1189	6930	40391	235416	...
C_n	**1**	**3**	17	99	577	3363	19601	114243	665857	...

The closed form formula for any number sequences is a very useful tool to derive identities. For balancing and Lucas-balancing numbers, the closed form formulas(Binet's formula) are, respectively, given[29] as

$$B_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} = \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{8}} \ and \ C_n = \frac{\alpha_1^n + \alpha_2^n}{2}, \tag{3}$$

where $\alpha_1 = 3 + \sqrt{8}$ and $\alpha_2 = 3 - \sqrt{8}$. And, also we have

$$\alpha_1 + \alpha_1 = 6, \alpha_1 - \alpha_1 = 2\sqrt{8} \ and \ \alpha_1\alpha_2 = 1.$$

Definition 1.1. [7] *A r-circulant matrix is defined as*

$$P[r] = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix},$$

where $r \in \mathbb{C} - \{0\}$. The r-circulant matrix $P[r]$ can be determined by the vector $\vec{a} = (c_0, c_1, c_2, ..., c_{n-2}, c_{n-1})$ where $\vec{a}$ is called the circulant vector.

Definition 1.2. *For $r = 1$ and $r = -1$, the r-circulant matrices are known as the right circulant matrices and the skew-right circulant matrices, respectively.*

The r-circulant matrices are the Toeplitz matrices[16] and they are diagonal-constant matrices.

Definition 1.3. *For any matrix $A = [a_{ij}]_{n\times n}$, the Euclidean norm of A is denoted by $||A||_E$ and defined as*

$$||A||_E = \sqrt{\sum_{i,j=0}^{n-1} |a_{ij}|^2}$$

and the spectral norm of A, denoted by $||A||_2$ is defined by

$$||A||_2 = max\{|\lambda_i|\},$$

where $i = 0, ..., n-1$ and $\lambda_i's$ are eigenvalues of A.

Definition 1.4. *The eigenvalues of r-circulant matrices P[r] are given as*

$$\lambda_t = \sum_{i=0}^{n-1} c_i(\rho\omega^{-t})^i$$

where ρ is the n-th root of r, ω is the n-th root of unity and $t = 0, 1, ..., n-1$.

Lemma 1.1. *By the virtue of* [7], *the Euclidean norm for r-circulant matrices P[r] is given by*

$$||P[r]||_E = \sqrt{\sum_{i=0}^{n-1} |c_i|^2[n - i(1 - |r|^2)]}. \tag{4}$$

Lemma 1.2. *For any x and y, we have $\prod_{i=0}^{n-1}(x - y\rho\omega^{-i}) = x^n - ry^n$.*

The following identities of the balancing numbers and Lucas-balancing numbers [25, 4] will be used to prove our main result.

Lemma 1.3. *For all integers m and n, we have*

(1) $B_n = \dfrac{\alpha_1^n - \alpha_2^n}{2\sqrt{8}}$ *and* $C_n = \dfrac{\alpha_1^n + \alpha_2^n}{2}$,
(2) $B_{m-n}B_{m+n} = (B_m - B_n)(B_m + B_n)$,
(3) $C_{n-m}C_{n+m} - C_n^2 = \frac{1}{2}(C_{2m} - 1)$,
(4) $C_{2n} = 2C_n^2 - 1$ *(if $m = n$ in (3)).*

2. Main results

Let us define r-circulant matrices $B[r] = [b_{ij}]_{i,j=1}^n$ and $C[r] = [c_{ij}]_{i,j=1}^n$ with balancing (B_n) and Lucas-balancing (C_n) numbers, respectively, as,

$$b_{ij} = \begin{cases} B_{s+(j-i)t} & : \ i \leq j \\ rB_{s+(n+j-i)t} & : \ i > j \end{cases} \quad \textit{and} \quad c_{ij} = \begin{cases} C_{s+(j-i)t} & : \ i \leq j \\ rC_{s+(n+j-i)t} & : \ i > j \end{cases}$$

i.e of the form

$$B[r] = \begin{bmatrix} B_s & B_{s+t} & B_{s+2t} & \dots & B_{s+(n-2)t} & B_{s+(n-1)t} \\ rB_{s+(n-1)t} & B_s & B_{s+t} & \dots & B_{s+(n-3)t} & B_{s+(n-2)t} \\ rB_{s+(n-2)t} & rB_{s+(n-1)t} & B_s & \dots & B_{s+(n-4)t} & B_{s+(n-3)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rB_{s+2t} & rB_{s+3t} & rB_{s+4t} & \dots & B_s & B_{s+t} \\ rB_{s+t} & rB_{s+2t} & rB_{s+3t} & \dots & rB_{s+(n-1)t} & B_s \end{bmatrix}_{n\times n}, \quad (5)$$

$$C[r] = \begin{bmatrix} C_s & C_{s+t} & C_{s+2t} & \dots & C_{s+(n-2)t} & C_{s+(n-1)t} \\ rC_{s+(n-1)t} & C_s & C_{s+t} & \dots & C_{s+(n-3)t} & C_{s+(n-2)t} \\ rC_{s+(n-2)t} & rC_{s+(n-1)t} & C_s & \dots & C_{s+(n-4)t} & C_{s+(n-3)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rC_{s+2t} & rC_{s+3t} & rC_{s+4t} & \dots & C_s & C_{s+t} \\ rC_{s+t} & rC_{s+2t} & rC_{s+3t} & \dots & rC_{s+(n-1)t} & C_s \end{bmatrix}_{n\times n}, \quad (6)$$

where $r \in \mathbb{C} - \{0\}$ and s, t are arbitrary integer.

In this article, we investigate the above r-circulant matrices $B[r]$ and $C[r]$ with entries balancing numbers(B_n) and Lucas-balancing numbers(C_n), respectively.

Our aim is to obtain the eigenvalues, determinants, Euclidean norms and the spectral norm for the matrices $B[r]$ and $C[r]$.

Lemma 2.1. *For integers s and t, we have*

$$G_B(s,t) = \frac{1}{2\sqrt{8}}(\alpha_1^s\alpha_2^t - \alpha_1^t\alpha_2^s) = \begin{cases} B_{s-t} & : s > t \\ 0 & : s = t \\ B_{t-s} & : s < t \end{cases}$$

and

$$G_C(s,t) = \frac{1}{2}(\alpha_1^s\alpha_2^t + \alpha_1^t\alpha_2^s) = \begin{cases} C_{s-t} & : s > t \\ 1 & : s = t \\ C_{t-s} & : s < t \end{cases}.$$

Proof. It can be easily proved using the relation $\alpha_1\alpha_2 = 1$ and (1) of Lemma 1.3. $\square$

In the following subsection, we obtain the eigenvalues for matrices $B[r]$ and $C[r]$.

2.1. Eigenvalues of B[r] and C[r].

Proposition 2.1. *The eigenvalues of r-circulant matrices $B[r]$ are*

$$\lambda_k(r) = \frac{B_s - rB_{s+nt} - (G_B(s,t) - rB_{s+(n-1)t})\rho\omega^{-k}}{(1-\alpha_1^t\rho\omega^{-k})(1-\alpha_2^t\rho\omega^{-k})},$$

where ρ is the n-th root of r, ω is the n-th root of unity and $k = 0, 1, 2, ..., n-1$.

Proof. From Definition 1.4, we have

$$\begin{aligned}
\lambda_k(r) &= \sum_{m=0}^{n-1} B_{s+mt}(\rho\omega^{-k})^m = \sum_{m=0}^{n-1}\left[\frac{\alpha_1^{s+mt}-\alpha_2^{s+mt}}{2\sqrt{8}}\right](\rho\omega^{-k})^m \text{ (using Lemma 1.3)}\\
&= \frac{1}{2\sqrt{8}}\left[\alpha_1^s\sum_{m=0}^{n-1}(\alpha_1^t\rho\omega^{-k})^m - \alpha_2^s\sum_{m=0}^{n-1}(\alpha_2^t\rho\omega^{-k})^m\right]\\
&= \frac{1}{2\sqrt{8}}\left[\alpha_1^s\left(\frac{1-(\alpha_1^t\rho\omega^{-k})^n}{1-\alpha_1^t\rho\omega^{-k}}\right) - \alpha_2^s\left(\frac{1-(\alpha_2^t\rho\omega^{-k})^n}{1-\alpha_2^t\rho\omega^{-k}}\right)\right]\\
&= \frac{1}{2\sqrt{8}}\left[\alpha_1^s\left(\frac{1-r\alpha_1^{nt}}{1-\alpha_1^t\rho\omega^{-k}}\right) - \alpha_2^s\left(\frac{1-r\alpha_2^{nt}}{1-\alpha_2^t\rho\omega^{-k}}\right)\right]\\
&= \frac{1}{2\sqrt{8}}\left[\left(\frac{\alpha_1^s-r\alpha_1^{s+nt}}{1-\alpha_1^t\rho\omega^{-k}}\right) - \left(\frac{\alpha_2^s-r\alpha_2^{s+nt}}{1-\alpha_2^t\rho\omega^{-k}}\right)\right]\\
&= \frac{1}{2\sqrt{8}}\Big(\frac{\alpha_1^s-r\alpha_1^{s+nt}-\alpha_2^s+r\alpha_2^{s+nt}-(\alpha_1^s\alpha_2^t-\alpha_1^t\alpha_2^s)\rho\omega^{-k}}{(1-\alpha_1^t\rho\omega^{-k})(1-\alpha_2^t\rho\omega^{-k})}\\
&\qquad + \frac{r(\alpha_1^{s+(n-1)t}-\alpha_2^{s+(n-1)t})\rho\omega^{-k}}{(1-\alpha_1^t\rho\omega^{-k})(1-\alpha_2^t\rho\omega^{-k})}\Big)\\
&= \frac{B_s-rB_{s+nt}-(G_B(s,t)-rB_{s+(n-1)t})\rho\omega^{-k}}{(1-\alpha_1^t\rho\omega^{-k})(1-\alpha_2^t\rho\omega^{-k})} \quad \text{(using (1) of Lemma 1.3)},
\end{aligned}$$

where $G_B(s,t) = \frac{1}{2\sqrt{8}}(\alpha_1^s\alpha_2^t - \alpha_1^t\alpha_2^s) = \begin{cases} B_{s-t} & : s>t \\ 0 & : s=t \\ B_{t-s} & : s<t \end{cases}.$

This completes the proof. □

For $r=1$ and $r=-1$ in the above proposition, we get the eigenvalues for the right circulant and the skew-right circulant balancing matrices respectively, given in the following corollary.

Corollary 2.1. *The eigenvalues of circulant matrices B[1] and B[-1] are, respectively,*

$$\lambda_k(B[1]) = \frac{B_s - B_{s+nt} - (G_B(s,t) - B_{s+(n-1)t})\omega^{-k}}{(1-\alpha_1^t\omega^{-k})(1-\alpha_2^t\omega^{-k})} \quad \textit{and} \tag{7}$$

$$\lambda_k(B[-1]) = \frac{B_s + B_{s+nt} - (G_B(s,t) + B_{s+(n-1)t})\tau\omega^{-k}}{(1-\tau\alpha_1^t\omega^{-k})(1-\tau\alpha_2^t\omega^{-k})}, \tag{8}$$

where τ is n-th root of -1 and $k = 0,1,2,...,n-1$.

Corollary 2.2. *For $k=0$ in eqn.(7), we have*

$$\sum_{m=0}^{n-1} B_{s+mt} = \frac{B_s - B_{s+nt} - G_B(s,t) + B_{s+(n-1)t}}{2(1-C_t)}.$$

Proof. Using $\alpha_1\alpha_2 = 1$ and (1) of Lemma 1.3, we write

$$\begin{aligned}
(1-\alpha_1^t)(1-\alpha_2^t) &= 1 + \alpha_1^t\alpha_2^t - (\alpha_1^t + \alpha_2^t)\\
&= 2 - 2C_t.
\end{aligned}$$

As required. □

Proposition 2.2. *The eigenvalues of r-circulant matrices $C[r]$ are given by*

$$\sigma_k(r) = \frac{C_s - rC_{s+nt} - (G_C(s,t) - rC_{s+(n-1)t})\rho\omega^{-k}}{(1-\alpha_1^t\rho\omega^{-k})(1-\alpha_2^t\rho\omega^{-k})},$$

where ρ is the n-th root of r, ω is the n-th root of unity and $k = 0, 1, 2, ..., n-1$.

Proof. The argument is similar to Proposition 2.1 where $G_C(s,t)$ is as defined in the Lemma 2.1. □

Here, the eigenvalues for the right circulant and the skew-right circulant Lucas-balancing matrices are given by setting $r = 1$ and $r = -1$ in the above proposition as follows.

Corollary 2.3. *The eigenvalues of circulant matrices C[1] and C[-1] are, respectively,*

$$\sigma_k(C[1]) = \frac{C_s - C_{s+nt} - (G_C(s,t) - C_{s+(n-1)t})\omega^{-k}}{(1-\alpha_1^t\omega^{-k})(1-\alpha_2^t\omega^{-k})} \quad \textit{and} \tag{9}$$

$$\sigma_k(C[-1]) = \frac{C_s + C_{s+nt} - (G_C(s,t) + C_{s+(n-1)t})\tau\omega^{-k}}{(1-\tau\alpha_1^t\omega^{-k})(1-\tau\alpha_2^t\omega^{-k})}, \tag{10}$$

where τ is n-th root of -1 and $k = 0, 1, 2, ..., n-1$.

Thus, for $k = 0$ in eqn.(9), we obtain the finite sum formula for Lucas balancing numbers having arithmetic indices as follows

$$\sum_{m=0}^{n-1} C_{s+mt} = \frac{C_s - C_{s+nt} - G_C(s,t) + C_{s+(n-1)t}}{2(1-C_t)}.$$

Now, we aim to obtain the determinant and norms of r-circulant balancing and Lucas-balancing matrices.

2.2. Determinants and norms of B[r] and C[r].

Theorem 2.1. *The determinants of r-circulant matrices $B[r]$ and $C[r]$ are given, respectively, by*

$$det(B[r]) = \frac{(B_s - rB_{s+nt})^n - r(G_B(s,t) - rB_{s+(n-1)t})^n}{1 + r^2 - 2rC_{nt}} \quad \textit{and} \tag{11}$$

$$det(C[r]) = \frac{(C_s - rC_{s+nt})^n - r(G_C(s,t) - rC_{s+(n-1)t})^n}{1 + r^2 - 2rC_{nt}}. \tag{12}$$

Proof. We establish the result using the fact that the determinant is a product of eigenvalues. Here,

$$\begin{aligned}
\prod_{k=0}^{n-1}\lambda_k(r) &= \prod_{k=0}^{n-1}\left(\frac{B_s - rB_{s+nt} - (G_B(s,t) - rB_{s+(n-1)t})\rho\omega^{-k}}{(1-\alpha_1^t\rho\omega^{-k})(1-\alpha_2^t\rho\omega^{-k})}\right) \\
&= \frac{(B_s - rB_{s+nt})^n - r(G_B(s,t) - rB_{s+(n-1)t})^n}{(1-r\alpha_1^{nt})(1-r\alpha_2^{nt})} \quad \text{(using Lemma 1.2)} \\
&= \frac{(B_s - rB_{s+nt})^n - r(G_B(s,t) - rB_{s+(n-1)t})^n}{1 + r^2 - 2rC_{nt}}.
\end{aligned}$$

For $det(C[r])$, the argument is the same as first part.
This completes the proof. □

As a special case of Theorem 2.1, for $r = 1$ and $r = -1$ the determinant of right circulant and skew-right circulant matrices have been obtained in the following corollary.

Corollary 2.4. *The determinant of matrices B[1] and B[-1] are, respectively,*

$$det(B[1]) = \frac{(B_s - B_{s+nt})^n - (G_B(s,t) - B_{s+(n-1)t})^n}{2(1-C_{nt})} \quad and$$
$$det(B[-1]) = \frac{(B_s + B_{s+nt})^n + (G_B(s,t) + B_{s+(n-1)t})^n}{2(1+C_{nt})}.$$

Corollary 2.5. *The determinant of matrices C[1] and C[-1] are, respectively,*

$$det(C[1]) = \frac{(C_s - C_{s+nt})^n - (G_C(s,t) - C_{s+(n-1)t})^n}{2(1-C_{nt})} \quad and$$
$$det(C[-1]) = \frac{(C_s + C_{s+nt})^n + (G_C(s,t) + C_{s+(n-1)t})^n}{2(1+C_{nt})}.$$

2.3. **Matrix norms.** In the case of r-circulant matrices $B[r]$ and $C[r]$, we have the following results.

The norm $||.||_1$ and $||.||_\infty$ for a matrix $A = [a_{ij}]_{m\times n}$ are given[18], respectively, by

$$||A||_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}| \text{ and } ||A||_\infty = \max_{1\le j\le m} \sum_{i=1}^{n} |a_{ij}|. \tag{13}$$

Theorem 2.2. *For balancing numbers B_n and Lucas-balancing numbers C_n, we have*

$$||B[r]||_1 = ||B[r]||_\infty = \begin{cases} B_s + |r|\sum_{m=1}^{n-1} B_{s+mt} & : if \ |r| > 1, \\ \sum_{m=0}^{n-1} B_{s+mt} & : if \ |r| \le 1, \end{cases}$$
$$||C[r]||_1 = ||C[r]||_\infty = \begin{cases} C_s + |r|\sum_{m=1}^{n-1} C_{s+mt} & : if \ |r| > 1, \\ \sum_{m=0}^{n-1} C_{s+mt} & : if \ |r| \le 1. \end{cases}$$

Now, in the following theorems, we establish the euclidean norm and bounds for the spectral norm.

Theorem 2.3. *The Euclidean norm of r-circulant matrices $B[r]$ is given by*

$$||B[r]||_E = \sqrt{\sum_{m=0}^{n-1} \left(\frac{C^2_{s+mt} - 1}{8}\right)[n - m(1-|r|^2)]}.$$

Proof. From Lemma 1.1, we have

$$\begin{aligned} ||B[r]||^2_E &= \sum_{m=0}^{n-1} |B_{s+mt}|^2[n - m(1-|r|^2)] \\ &= \sum_{m=0}^{n-1} \left(\frac{\alpha_1^{s+mt} - \alpha_2^{s+mt}}{2\sqrt{8}}\right)^2 [n - m(1-|r|^2)] \quad \text{(using (1) of Lemma 1.3)} \\ &= \sum_{m=0}^{n-1} \left(\frac{\alpha_1^{2(s+mt)} + \alpha_2^{2(s+mt)} - 2}{32}\right)[n - m(1-|r|^2)] \\ &= \sum_{m=0}^{n-1} \left(\frac{C_{2(s+mt)} - 1}{16}\right)[n - m(1-|r|^2)] \\ &= \sum_{m=0}^{n-1} \left(\frac{2C^2_{s+mt} - 2}{16}\right)[n - m(1-|r|^2)] \quad \text{(using (4) of Lemma 1.3).} \end{aligned}$$

This completes the proof. □

Special case of above theorem for $r = \pm 1$ gives the Euclidean norm of the right circulant and the skew right circulant matrices with balancing numbers as

$$||B[\pm 1]||_E = \sqrt{n \sum_{m=0}^{n-1} \left(\frac{C^2_{s+mt} - 1}{8} \right)}.$$

Theorem 2.4. *The Euclidean norm of r-circulant matrices C[r] is given by*

$$||C[r]||_E = \sqrt{\sum_{m=0}^{n-1} \left(\frac{C_{2(s+mt)} + 1}{2} \right) [n - m(1 - |r|^2)]}.$$

Proof. The argument is the same as the previous theorem. □

Corollary 2.6. *The Euclidean norm of right circulant and skew right circulant matrices with Lucas-balancing numbers are given by*

$$||C[\pm 1]||_E = \sqrt{n \sum_{m=0}^{n-1} C^2_{s+mt}}.$$

Proof. Setting $r = \pm 1$ in the Theorem 2.4, we get

$$\begin{aligned} ||C[\pm 1]||_E &= \sqrt{\sum_{m=0}^{n-1} \left(\frac{C_{2(s+mt)} + 1}{2} \right) n} \\ &= \sqrt{\frac{n}{2} \sum_{m=0}^{n-1} \left(C_{2(s+mt)} + 1 \right)} \\ &= \sqrt{n \sum_{m=0}^{n-1} C^2_{s+mt}} \qquad \text{(using (4) of Lemma 1.3).} \end{aligned}$$

As required. □

2.3.1. *Bounds for the spectral norms.*

Theorem 2.5. *Bounds for the spectral norms of r-circulant matrices B[r] are given by*

$$||B[r]||_2 \leq \frac{|B_s| + |r||B_{s+nt}| + |r|^{\frac{1}{n}} |G_B(s,t)| + |r|^{\frac{1}{n}+1} |B_{s+(n-1)t}|}{|\xi(\rho)|^2}, \tag{14}$$

where $\xi(\rho) = min\{|1 - \alpha_1^t \rho \omega^{-k}|, |1 - \alpha_2^t \rho \omega^{-k}|\}$.

Proof. From Proposition 2.1, for $k = 0, 1, 2, ..., n-1$, we write

$$\begin{aligned} |\lambda_k(r)| &= \left| \frac{B_s - rB_{s+nt} - (G_B(s,t) - rB_{s+(n-1)t})\rho\omega^{-k}}{(1 - \alpha_1^t \rho \omega^{-k})(1 - \alpha_2^t \rho \omega^{-k})} \right| \\ &\leq \frac{|B_s| + |r||B_{s+nt}| + |r|^{\frac{1}{n}} (|G_B(s,t)| + |r||B_{s+(n-1)t}|)}{|(1 - \alpha_1^t \rho \omega^{-k})||(1 - \alpha_2^t \rho \omega^{-k})|} \\ &\leq \frac{|B_s| + |r||B_{s+nt}| + |r|^{\frac{1}{n}} |G_B(s,t)| + |r|^{\frac{1}{n}+1} |B_{s+(n-1)t}|}{|\xi(\rho)|^2} \end{aligned}$$

where $\xi(\rho) = min\{|1-\alpha_1^t\rho\omega^{-k}|, |1-\alpha_2^t\rho\omega^{-k}|\}$. Thus, we can conclude that

$$||B[r]||_2 \leq \frac{|B_s| + |r||B_{s+nt}| + |r|^{\frac{1}{n}}|G_B(s,t)| + |r|^{\frac{1}{n}+1}|B_{s+(n-1)t}|}{|\xi(\rho)|^2}.$$

As required. □

Theorem 2.6. *Bounds for the spectral norms of r-circulant matrices $C[r]$ are given by*

$$||C[r]||_2 \leq \frac{|C_s| + |r||C_{s+nt}| + |r|^{\frac{1}{n}}|G_C(s,t)| + |r|^{\frac{1}{n}+1}|C_{s+(n-1)t}|}{|\xi(\rho)|^2}, \tag{15}$$

where $\xi(\rho) = min\{|1-\alpha_1^t\rho\omega^{-k}|, |1-\alpha_2^t\rho\omega^{-k}|\}$.

Proof. The argument is similar to the above theorem, where $G_C(s,t)$ is as defined in the Lemma 2.1. □

As a special case of above theorems, we have the following corollary.

Corollary 2.7. *Bounds for spectral norms of right circulant matrices B[1] & C[1] and skew-right circulant matrices B[-1] & C[-1] are respectively*

$$||B[1]||_2 \leq \frac{|B_s| + |B_{s+nt}| + |G_B(s,t)| + |B_{s+(n-1)t}|}{|\xi(\omega)|^2}, \tag{16}$$

$$||B[-1]||_2 \leq \frac{|B_s| + |B_{s+nt}| + |G_B(s,t)| + |B_{s+(n-1)t}|}{|\xi(\tau)|^2}, \tag{17}$$

$$||C[1]||_2 \leq \frac{|C_s| + |C_{s+nt}| + |G_C(s,t)| + |C_{s+(n-1)t}|}{|\xi(\omega)|^2}, \tag{18}$$

$$||C[-1]||_2 \leq \frac{|C_s| + |C_{s+nt}| + |G_C(s,t)| + |C_{s+(n-1)t}|}{|\xi(\tau)|^2}, \tag{19}$$

where $\xi(\omega) = min\{|1-\alpha_1^t\omega^{-k}|, |1-\alpha_2^t\omega^{-k}|\}$, $\xi(\tau) = min\{|1-\alpha_1^t\tau\omega^{-k}|, |1-\alpha_2^t\tau\omega^{-k}|\}$ and $G_B(s,t), G_C(s,t)$ are as defined in the Lemma 2.1.

Theorem 2.7. *If $r \in \mathbb{Z}$, then $(B_s - rB_{s+nt})^n - r(G_B(s,t) - rB_{s+(n-1)t})^n$ and $(C_s - rC_{s+nt})^n - r(G_C(s,t) - rC_{s+(n-1)t})^n$ are multiple of $1 + r^2 - 2rC_{nt}$.*

Proof. Since, the entries of matrices $B[r]$ and $C[r]$ are integers and we know that the determinant of matrices whose entries are integers is an integer. So $det(B[r]), det(C[r]) \in \mathbb{Z}$. Hence

$$\frac{(B_s - rB_{s+nt})^n - r(G_B(s,t) - rB_{s+(n-1)t})^n}{1 + r^2 - 2rC_{nt}} \in \mathbb{Z}$$

and

$$\frac{(C_s - rC_{s+nt})^n - r(G_C(s,t) - rC_{s+(n-1)t})^n}{1 + r^2 - 2rC_{nt}} \in \mathbb{Z}.$$

It gives that $(B_s - rB_{s+nt})^n - r(G_B(s,t) - rB_{s+(n-1)t})^n$ and $(C_s - rC_{s+nt})^n - r(G_C(s,t) - rC_{s+(n-1)t})^n$ are multiple of $1 + r^2 - 2rC_{nt}$. □

By setting $r = 1$ and $r = -1$ in the above theorem, we have the following corollary.

Corollary 2.8. *For balancing numbers B_n and Lucas-balancing numbers C_n, we have*

(1) $(B_s - B_{s+nt})^n - (G_B(s,t) - B_{s+(n-1)t})^n$ *and* $(C_s - C_{s+nt})^n - (G_C(s,t) - C_{s+(n-1)t})^n$ *are multiple of* $2(1 - C_{nt})$.

(2) $(B_s + B_{s+nt})^n + (G_B(s,t) + B_{s+(n-1)t})^n$ *and* $(C_s + C_{s+nt})^n + (G_C(s,t) + C_{s+(n-1)t})^n$ *are multiple of* $2(1 + C_{nt})$.

3. Conclusion

In summary, here we have studied the r-circulant matrices $B[r]$ and $C[r]$ involving the balancing and Lucas-balancing numbers respectively with arithmetic indices. For matrices $B[r]$ and $C[r]$, we have established the direct formula for the eigenvalues, the determinant, the Euclidean norm and the bound for the spectral norm. Furthermore, we have extended the concept to right circulant matrices and skew-right circulant matrices and established all the above results. Lastly, we have obtained some interesting results and identities involving sum identities and divisibility.

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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