TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 787-800

# ALGEBRAIC SUM AND ALGEBRAIC PRODUCT OF SPHERICAL NEUTROSOPHIC SETS

A. MANIMARAN[1], V. MUTHUKUMARAN[2], R. VIJAYKRISHNARAJ[3], I. SILAMBARASAN[4*], §

Abstract. The Spherical Neutrosophic Set is a generalization of Spherical Fuzzy Set. In this paper, we define Algebraic sum and Algebraic product of spherical neutrosophic sets (SNSs) and investigate their desirable properties. Further, we construct scalar multiplication $(nA)$ and exponentiation $(A^n)$ operations of a spherical neutrosophic sets. Finally, define a new operation(@) on spherical neutrosophic sets and discuss distributive law in the case where the operations of $\oplus, \otimes, \wedge$ and $\vee$ are combined each other.

Significant Statement: The important methodological problems of other analyzed theories were revealed and discussed. Many of the novel theories, such as Neutrosophic, Pythagorean, Spherical and Picture sets theories, were developed to solve interior problems of the Atanassov's intuitionistic fuzzy sets. Since the results obtained within the framework of these criticized novel theories appeared to be significantly worse than expected, in this paper, an alternative approach based on the extension of the Spherical fuzzy Sets in the framework of the Spherical Neutrosophic Sets theory is proposed. Our study is actually the study of an important type of advanced fuzzy algebraic structure.

Keywords: Intuitionistic fuzzy set, Pythagorean fuzzy set, Picture fuzzy set. Spherical fuzzy set, Spherical Neutrosophic set, Algebraic sum, Algebraic product, Scalar multiplication, Exponentiation operations.

AMS Subject Classification: 03B99, 08A72, 03E99.

## 1. Introduction

Fuzzy set theory is one of the most powerful track for treating the multi-attribute decision making problems. Based on fuzziness circumstances fuzzy sets (FSs), developed by Zadeh [27], was initially used. In FSs each element x of the domain set contains only one

[1] Department of Mathematics, School of Advanced Sciences, VIT, Vellore, 632014, Tamilnadu, India.
e-mail: marans2011@gmail.com; ORCID: https://orcid.org/0000-0001-6717-1152.
[2] Department of Mathematics, FEAT, SRM Institute of Science and Technology, 603203, Kattankulathur, India.
e-mail: muthu.v2404@gmail.com; ORCID: https://orcid.org/0000-0002-3393-5596.
[3] Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, India.
e-mail: r.vijay08@gmail.com; ORCID: https://orcid.org/0000-0001-9540-729X.
[4] Department of Mathematics, Annamalai University, 608002, Chidambaram, India.
e-mail: sksimbuking@gmail.com; ORCID: https://orcid.org/0000-0002-7437-4043.
* Corresponding author.
§ Manuscript received: May 10, 2022; accepted: August 01, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.2 

index namely as degree of membership $\zeta_A(\hat{x})$ which oscillate from 0 to 1. Non membership degree for the FS is straightforward equivalent to $1 - \zeta_A(\hat{x})$. However, sometime FS has some drawbacks for example, it has no ability to show the neutral state (which neither favor nor disfavor). However, Intuitionistic fuzzy set (IFS) developed by Atanassov [1], to apprehend the uncertainties or in exact information about degree of membership. Atanassovs IFSs are the generalization of Zadehs FSs.

Since in some real life decision theory the decision makers deal with the situation of particular attributes where values of their summation of membership degrees exceeds 1. In such condition, IFSs has no ability to obtain any satisfactory result. To overcome this situation Yager [25] developed the idea of Pythagorean fuzzy set (PyFS) as a generalization of IFS, which satisfies that the value of square summation of its membership degrees is less then or equals to 1. Now the situation where the neutral membership degree calculate independently in real life problems, the IFS and PyFS fail to attain any satisfactory result. Based on these circumstances, to overcome this situation, Cuong [2] initiated the idea of picture fuzzy set (PFS). He utilized three index (membership degree $\zeta_A(\hat{x})$, neutral-membership degree , and $\eta_A(\breve{x})$ non-membership degree $\delta_A(\hat{x})$) in PFS with the condition that is $0 \leq \zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x}) \leq 1$. Obviously PFSs is more suitable than IFS and PyFS to deal with fuzziness and vagueness. A lot of research works on picture fuzzy set based on algebraic structures were done by (Dogra and Pal, [4]-[9]). In the recent years, the idea of Pythagorean fuzzy set and Picture fuzzy set has been applied effectively by [11, 14, 15, 16, 26, 28].

Sometimes in real life, we face many problems which cannot be handled by using PFS for example when $\zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x}) > 1$. In such condition, PFS has no ability to obtain any satisfactory result. To state this condition, we give an example: for support and against the degree of membership of an alternative are $0.2, 0.6$ and $0.6$ respectively. This satisfies the condition that their sum exceeds 1 and are not presented for PFS. Based on these circumstances, the idea of spherical fuzzy sets (SFSs) is introduced as a generalization of PFS. In SFS, membership degrees are gratifying the condition. $0 \leq \zeta_A^2(\hat{x}) + \eta_A^2(\breve{x}) + \delta_A^2(\hat{x}) \leq 1$.

Neutrosophic sets (NSs) proposed by (Smarandache, [17]- [21]) which is a generalization of fuzzy sets and intuitionistic fuzzy set, is a powerful tool to deal with incomplete, indeterminate and inconsistent information which exist in the real world. Neutrosophic sets are characterized by truth membership function (T), indeterminacy membership function (I) and falsity membership function (F). This theory is very important in many application areas since indeterminacy is quantified explicitly and the truth membership function, indeterminacy membership function and falsity membership functions are independent. Wang, Smarandache, Zhang, & Sunderraman [24] introduced the concept of single valued neutrosophic set. All the factors described by the single-valued neutrosophic set are very suitable for human thinking due to the imperfection of knowledge that human receives or observes from the external world.

Spherical Neutrosophic Set (SNS) was introduced by Smarandache [20] in 2017 The Spherical Neutrosophic Set is a generalization of Spherical Fuzzy Set, because we may restrain the SNSs components to the unit interval $\zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x}) \leq [0, 1]$, and the sum of the squared components to 1, i.e. $0 \leq \zeta_A^2(\hat{x}) + \eta_A^2(\breve{x}) + \delta_A^2(\hat{x}) \leq 1$ . Further on, if replacing $\eta_A(\breve{x}) = 0$ into the Spherical Fuzzy Set, we obtain as particular case the Pythagorean Fuzzy Set. Kandasamy and Smarandache [13] introduced fuzzy relational maps and neutrosophic relational maps.In the recent years, the idea of neutrosophic set has been applied effectively by [3, 10, 22, 23].

The motivation of introducing SFS that in the real-life decision process, the sum of the support (membership) degree and the against (nonmembership) degree to which an alternative satisfying a criterion provided by the decision maker may be larger than 1 but their square sum is equal to or less than 3.

In this paper, we develope the Spherical fuzzy sets to Spherical Neutrosophic Sets.

This paper is organized as follows. In section 2, we recall some preliminary definitions regarding the topic. In section 3, Algebraic sum and Algebraic product of spherical neutrosophic sets and investigates some algebraic properties such as idempotency, commutativity, associativity, absorption distributivity, and De Morgan's laws over complement.. In section 4, define a new operation(@) on spherical neutrosophic sets and investigates their desirable properties. We write the conclusion of the paper in section 5.

## 2. Preliminary Definitions

Here we recall some preliminary definitions regarding the topic.

**Definition 2.1.** [1] *A intuitionistic fuzzy set $A$ on a universe $X$ is an object of the form*

$$A = \{(x, \zeta_A(\hat{x}), \delta_A(\hat{x}))\,|x \in X\}$$

*where $\zeta_A(\hat{x}) \in [0,1]$ is called the degree of membership of $x$ in $A$, $\delta_A(\hat{x}) \in [0,1]$ is called the degree of non-membership of $x$ in $A$, and where $\zeta_A(\hat{x})$ and $\delta_A(\hat{x})$ satisfy the following condition:*

$$0 \leq \zeta_A(\hat{x}) + \delta_A(\hat{x}) \leq 1 \textit{ for all } x \in X$$

**Definition 2.2.** [24] *Let $X$ be a universal set. Then, a Pythagorean fuzzy set (PyFS) $A$, which is a set of ordered pairs over $X$, is defined by the following:*

$$A = \{(x, \zeta_A(\hat{x}), \delta_A(\hat{x}))\,|x \in X\}$$

*where the functions $\zeta_A(\hat{x}) \in [0,1]$ and $\delta_A(\hat{x}) \in [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to $A$, which is a subset of $X$, and for every $x \in X$*

$$0 \leq \zeta_A^2(\hat{x}) + \delta_A^2(\hat{x}) \leq 1 \textit{ for all } x \in X$$

**Definition 2.3.** [2] *A picture fuzzy set (PFS) $A$ over the universe $X$ is defined as*

$$A = \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}))\,|x \in X\}$$

*where $\zeta_A(\hat{x}) \in [0,1]$ is called the degree of positive membership of $x \in A$, $\eta_A(\breve{x}) \in [0,1]$ is called the degree of neutral membership of $x$ in $A$ and $\delta_A(\hat{x}) \in [0,1]$ is called the degree of negative membership of $x$ in $A$, and where $\zeta_A(\hat{x})$, $\eta_A(\breve{x})$ and $\delta_A(\hat{x})$ satisfy the following condition:*

$$0 \leq \zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x}) \leq 1 \textit{ for all } x \in X$$

**Definition 2.4.** [12] *A spherical fuzzy set (SFS) $A$ over the universe $X$ is defined as*

$$A = \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}))\,|x \in X\}$$

*where $\zeta_A(\hat{x}) \in [0,1]$ is called the degree of positive membership of $x \in A$, $\eta_A(\breve{x}) \in [0,1]$ is called the degree of neutral membership of $x$ in $A$ and $\delta_A(\hat{x}) \in [0,1]$ is called the degree of negative membership of $x$ in $A$, and where $\zeta_A(\hat{x})$, $\eta_A(\breve{x})$ and $\delta_A(\hat{x})$ satisfy the following condition:*

$$0 \leq \zeta_A^2(\hat{x}) + \eta_A^2(\breve{x}) + \delta_A^2(\hat{x}) \leq 1 \textit{ for all } x \in X$$

**Definition 2.5.** [20] *A Neutrosophic fuzzy set $A$ on a universe $X$ is an object of the form*

$$A = \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}))\,|x \in X\}$$

*where* $\zeta_A(\hat{x}) \in [0,1]$ *is called the degree of positive membership of* $x$ *in* $A$, $\eta_A(\breve{x}) \in [0,1]$ *is called the degree of neutral membership of* $x$ *in* $A$ *and* $\delta_A(\hat{x}) \in [0,1]$ *is called the degree of negative membership of* $x$ *in* $A$, *and where* $\zeta_A(\hat{x})$, $\eta_A(\breve{x})$ *and* $\delta_A(\hat{x})$ *satisfy the following condition:*

$$0 \leq \zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x}) \leq 3 \text{ for all } x \in X$$

**Definition 2.6.** [23] *A Spherical neutrosophic set A of the form,* $A = \langle \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}) \rangle$ *of a non negative real numbers* $\zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}) \in [0,1]$ *satisfying the condition*

$$0 \leq \zeta_A^2(\hat{x}) + \eta_A^2(\hat{x}) + \delta_A^2(\hat{x}) \leq 3$$

*for all* $i, j$. *Where* $\zeta_A(\hat{x}) \in [0,1]$ *is called the degree of membership,* $\eta_A(\breve{x}) \in [0,1]$ *is called the degree of neutral membership and* $\delta_A(\hat{x}) \in [0,1]$ *is called the degree of non-membership.*

We denote the set of all SNSs over X by SNS(X).

## 3. Spherical neutrosophic sets and their basic operations

This section, we define Algebraic sum and Algebraic product of spherical neutrosophic sets and investigates some algebraic properties such as idempotency, commutativity, associativity, absorption distributivity, and De Morgan's laws over complement.

Now, we are going to define algebraic operations of Spherical neutrosophic sets by restricting the measure of positive, neutral and negative membership but keeping their sum in the interval $[0, \sqrt[2]{3}]$.

**Definition 3.1.** *If* $A$ *and* $B$ *are two spherical neutrosophic sets, then*
- $A < B$ *iff* $\forall x \in X, \zeta_A(\hat{x}) \leq \zeta_B(\hat{x}), \eta_A(\breve{x}) \leq \eta_B(\breve{x})$ *or* $\eta_A(\breve{x}) \geq \eta_B(\breve{x}), \delta_A(\hat{x}) \geq \delta_B(\hat{x})$
- $A^C = \{(x, \delta_A(\hat{x}), \eta_A(\breve{x}), \zeta_A(\hat{x})) \,|x \in X\}$
- $A \vee B = \{(x, \max(\zeta_A(\hat{x}), \zeta_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \min(\delta_A(\hat{x}), \delta_B(\hat{x}))) \,|x \in X\}$
- $A \wedge B = \{(x, \min(\zeta_A(\hat{x}), \zeta_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \max(\delta_A(\hat{x}), \delta_B(\hat{x}))) \,|x \in X\}$
- $A \oplus B = \left\{\left(x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\right) |x \in X\right\}$
- $A \otimes B$ $= \left\{\left(x, \zeta_A(\hat{x})\zeta_B(\hat{x}), \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}, \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}\right) |x \in X\right\}$.

**Example 3.1.** *Each element in an PFS is expressed by an ordered pair* $\langle \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}) \rangle$ *with* $\zeta_A(\hat{x}), \eta_A(\breve{x})$ *and* $\delta_A(\hat{x}) \in [0,1]$ *and* $0 \leq \zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x}) \leq 1$. *It was clearly seen that* $0.8+0.8+0.8 > 1$, *and thus it could not be described by PFS and SFS. To describe such evaluation in this paper we have proposed spherical neutrosophic set (SNS)and its algebraic operations. Each element in an SNS is expressed by an ordered pair* $\langle \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}) \rangle$ *with* $\zeta_A(\hat{x}), \eta_A(\breve{x})$ *and* $\delta_A(\hat{x}) \in [0,1]$ *and* $0 \leq \zeta_A^2(\hat{x}) + \eta_A^2(\hat{x}) + \delta_A^2(\hat{x}) \leq 3$. *Also, we can get* $(0.8)^2 + (0.8)^2 + (0.8)^2 = 0.64 + 0.64 + 0.64 = 1.92 \leq 3$, *which is good enough to apply the SNS to control it.*

**Definition 3.2.** *The scalar multiplication operation over SNS A and is defined by*

$$nA = \left\{\left(x, \sqrt{1 - [1 - \zeta_A^2(\hat{x})]^n}, [\eta_A(\breve{x})]^n, [\delta_A(\hat{x})]^n\right) |x \in X\right\}$$

**Definition 3.3.** *The exponentiation operation over SNS A and is defined by*

$$A^n = \left\{\left(x, [\zeta_A(\hat{x})]^n, \sqrt{1 - [1 - \eta_A^2(\hat{x})]^n}, \sqrt{1 - [1 - \delta_A^2(\hat{x})]^n}\right) |x \in X\right\}.$$

The following theorem relation between algebraic sum, and algebraic product of SNSs.

**Theorem 3.1.** *If* $A, B \in SNS(X)$, *then* $A \otimes B \leq A \oplus B$.

*Proof.* Let $A \oplus B$
$= \left\{ \left( x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x}) \right) |x \in X \right\}$ and
$A \otimes B$
$= \left\{ \left( x, \zeta_A(\hat{x})\zeta_B(\hat{x}), \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}, \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})} \right) |x \in X \right\}$
Assume that,

$$\zeta_A(\hat{x})\zeta_B(\hat{x}) \leq \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}$$

$(i.e)\qquad \zeta_A(\hat{x})\zeta_B(\hat{x}) - \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})} \geq 0$
$(i.e)\qquad \zeta_A^2(\hat{x})(1 - \zeta_B^2(\hat{x})) + \zeta_B^2(\hat{x})(1 - \zeta_A^2(\hat{x})) \geq 0$
which is true as $0 \leq \zeta_A^2(\hat{x}) \leq 1$ and $0 \leq \zeta_B^2(\hat{x}) \leq 1$
And

$$\eta_A(\breve{x})\eta_B(\breve{x}) \leq \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}$$

$(i.e)\qquad \eta_A(\breve{x})\eta_B(\breve{x}) - \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})} \geq 0$
$(i.e)\qquad \eta_A^2(\hat{x})(1 - \eta_B^2(\hat{x})) + \eta_B^2(\hat{x})(1 - \eta_A^2(\hat{x})) \geq 0$
which is true as $0 \leq \eta_A^2(\hat{x}) \leq 1$ and $0 \leq \eta_B^2(\hat{x}) \leq 1$
And

$$\delta_A(\hat{x})\delta_B(\hat{x}) \leq \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}$$

$(i.e)\qquad \delta_A(\hat{x})\delta_B(\hat{x}) - \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})} \geq 0$
$(i.e)\qquad \delta_A^2(\hat{x})(1 - \delta_B^2(\hat{x})) + \delta_B^2(\hat{x})(1 - \delta_A^2(\hat{x})) \geq 0$
which is true as $0 \leq \delta_A^2(\hat{x}) \leq 1$ and $0 \leq \delta_B^2(\hat{x}) \leq 1$
Hence $A \otimes B \leq A \oplus B$. □

**Theorem 3.2.** *For any spherical neutrosophic set A,*
$(i)$ $A \oplus A \geq A$,
$(ii)$ $A \otimes A \leq A$.

*Proof.* $(i)$ Let $A \oplus A$
$= \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x})) \,|x \in X\} \oplus \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x})) \,|x \in X\}$
$A \oplus A = \left\{ \left( x, \sqrt{2\zeta_A(\hat{x}) - (\zeta_A(\hat{x}))^2}, (\eta_A(\breve{x}))^2, (\delta_A(\hat{x}))^2 \right) |x \in X \right\}$
$\sqrt{2\zeta_A(\hat{x}) - (\zeta_A(\hat{x}))^2} = \sqrt{\zeta_A(\hat{x}) + \zeta_A(\hat{x})(1 - \zeta_A(\hat{x}))} \geq \zeta_A(\hat{x})$ for all $x \in X$
and $\quad (\eta_A(\breve{x}))^2 \leq \eta_A(\breve{x})$ for all $x \in X$
and $\quad (\delta_A(\hat{x}))^2 \leq \delta_A(\hat{x})$ for all $x \in X$
Hence $A \oplus A \geq A$.
Similarly, we can prove that (ii) $A \otimes A \leq A$. □

**Theorem 3.3.** *If $A, B, C \in SNS(X)$, then*
$(i)$ $A \oplus B = B \oplus A$,
$(ii)$ $A \otimes B = B \otimes B$,
$(iii)$ $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
$(iv)$ $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

*Proof.* $(i)$ Let $A \oplus B$
$= \left\{ \left( x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x}) \right) |x \in X \right\}$
$= \left\{ \left( x, \sqrt{\zeta_B^2(\hat{x}) + \zeta_A^2(\hat{x}) - \zeta_B^2(\hat{x})\zeta_A^2(\hat{x})}, \eta_B(\breve{x})\eta_A(\breve{x}), \delta_B(\hat{x})\delta_A(\hat{x}) \right) |x \in X \right\}$
$= B \oplus A$.
$(ii)$ Let $A \otimes B, \forall x \in X$.

$= \left(\zeta_A(\hat{x})\zeta_B(\hat{x}), \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}, \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}\right)$
$= \left(\zeta_B(\hat{x})\zeta_A(\hat{x}), \sqrt{\eta_B^2(\hat{x}) + \eta_A^2(\hat{x}) - \eta_B^2(\hat{x})\eta_A^2(\hat{x})}, \sqrt{\delta_B^2(\hat{x}) + \delta_A^2(\hat{x}) - \delta_B^2(\hat{x})\delta_A^2(\hat{x})}\right)$
$= B \otimes A.$
$(iii)$ Let $(A \oplus B) \oplus C$
$= \left(\left(\sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\right) \oplus \left(\zeta_{c_{ij}}, \eta_C(\breve{x}), \delta_{c_{ij}}\right)\right)$

$$= \Big[x, \sqrt{\left(\sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}\right)^2 + \zeta_{c_{ij}}^2 - \left(\sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}\right)^2 \zeta_{c_{ij}}^2},$$
$$\eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\delta_{c_{ij}} | x \in X\Big]$$
$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) + \zeta_{c_{ij}}^2 - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})\zeta_{c_{ij}}^2 - \zeta_A^2(\hat{x})\zeta_{c_{ij}}^2 - \zeta_B^2(\hat{x})\zeta_{c_{ij}}^2 + \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})\zeta_{c_{ij}}^2},$$
$$\eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\delta_{c_{ij}} | x \in X\Big]$$
$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) + \zeta_{c_{ij}}^2 - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_{c_{ij}}^2 - \zeta_B^2(\hat{x})\zeta_{c_{ij}}^2 + \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})\zeta_{c_{ij}}^2},$$
$$\eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\delta_{c_{ij}} | x \in X\Big]$$

Let $A \oplus (B \oplus C)$

$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \left(\sqrt{\zeta_B^2(\hat{x}) + \zeta_{c_{ij}}^2 - \zeta_B^2(\hat{x})\zeta_{c_{ij}}^2}\right)^2 - \zeta_A^2(\hat{x})\left(\sqrt{\zeta_B^2(\hat{x}) + \zeta_{c_{ij}}^2 - \zeta_B^2(\hat{x})\zeta_{c_{ij}}^2}\right)^2},$$
$$\eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\delta_{c_{ij}} | x \in X\Big]$$
$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) + \zeta_{c_{ij}}^2 - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_{c_{ij}}^2 - \zeta_B^2(\hat{x})\zeta_{c_{ij}}^2 + \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})\zeta_{c_{ij}}^2},$$
$$\eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\delta_{c_{ij}} | x \in X\Big]$$

Hence $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
Similarly, we can prove that (iv)$(A \otimes B) \otimes C = A \otimes (B \otimes C)$. □

**Theorem 3.4.** *If $A, B \in SNS(X)$, then*
$(i)$ $A \oplus (A \otimes B) \geq A$,
$(ii)$ $A \otimes (A \oplus B) \leq A$.

*Proof.* $(i)$ Let $A \oplus (A \otimes B)$
$= \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x})) | x \in X\} \oplus$
$\left\{\left(x, \zeta_A(\hat{x})\zeta_B(\hat{x}), \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}, \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}\right) | x \in X\right\}$

$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_A^2(\hat{x})\zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})[\zeta_A^2(\hat{x})\zeta_B^2(\hat{x})]}, \eta_A(\breve{x})[\sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}],$$
$$\delta_A(\hat{x})[\sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}] | x \in X\Big]$$
$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})[1 - \zeta_A^2(\hat{x})]}, \eta_A(\breve{x})\left(\sqrt{1 - [1 - \eta_A^2(\hat{x})][1 - \eta_B^2(\hat{x})]}\right),$$
$$\delta_A(\hat{x})\left(\sqrt{1 - [1 - \delta_A^2(\hat{x})][1 - \delta_B^2(\hat{x})]}\right) | x \in X\Big]$$

$\geq A.$
Hence $A \oplus (A \otimes B) \geq A$.
Similarly, we can prove that $(ii) A \otimes (A \oplus B) \leq A$. □

The following theorem is obvious.

**Theorem 3.5.** *If $A, B \in SNS(X)$, then*
$(i)$ $A \vee B = B \vee A$,
$(ii)$ $A \wedge B = B \wedge A$,

**Theorem 3.6.** *If $A, B, C \in SNS(X)$, then*
$(i)$ $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$,
$(ii)$ $A \otimes (B \vee C) = (A \otimes B) \vee (A \otimes C)$,
$(iii)$ $A \oplus (B \wedge C) = (A \oplus B) \wedge (A \oplus C)$,
$(iv)$ $A \otimes (B \wedge C) = (A \otimes B) \wedge (A \otimes C)$.

*Proof.* In the following, we shall prove $(i)$, and $(ii) - (iv)$ can be proved analogously.
$(i)$ Let $A \oplus (B \vee C)$

$$= \Big[x, \sqrt{\zeta_A^2(\hat{x}) + \max\left(\zeta_B^2(\hat{x}), \zeta_{c_{ij}}^2\right) - \zeta_A^2(\hat{x}).\max\left(\zeta_B^2(\hat{x}), \zeta_{c_{ij}}^2\right)},$$
$$\eta_A(\breve{x}).\max\left(\eta_B(\breve{x}), \eta_C(\breve{x})\right), \delta_A(\hat{x}).\max\left(\delta_B(\hat{x}), \delta_{c_{ij}}\right) | x \in X\Big]$$
$$= \Big[x, \sqrt{\max\left(\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}), \zeta_{a_{ij}^2} + \zeta_{c_{ij}}^2\right) - \max\left(\zeta_A^2(\hat{x})\zeta_B^2(\hat{x}), \zeta_A^2(\hat{x})\zeta_{c_{ij}}^2\right)},$$
$$\min\left(\eta_A(\breve{x})\eta_B(\breve{x}), \eta_A(\breve{x})\eta_C(\breve{x})\right), \min\left(\delta_A(\hat{x})\delta_B(\hat{x}), \delta_A(\hat{x})\delta_{c_{ij}}\right) | x \in X\Big]$$
$$= \Big[x, \sqrt{\max\left(\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x}), \zeta_A^2(\hat{x}) + \zeta_{c_{ij}}^2 - \zeta_A^2(\hat{x})\zeta_{c_{ij}}^2\right)},$$
$$\min\left(\eta_A(\breve{x})\eta_B(\breve{x}), \eta_A(\breve{x})\eta_C(\breve{x})\right), \min\left(\delta_A(\hat{x})\delta_B(\hat{x}), \delta_A(\hat{x})\delta_{c_{ij}}\right) | x \in X\Big]$$
$$= (A \oplus B) \vee (A \oplus C).$$ □

**Theorem 3.7.** *If $A, B \in SNS(X)$, then*
$(i)$ $(A \wedge B) \oplus (A \vee B) = A \oplus B$,
$(ii)$ $(A \wedge B) \otimes (A \vee B) = A \otimes B$,
$(iii)$ $(A \oplus B) \wedge (A \otimes B) = A \otimes B$,
$(iv)$ $(A \oplus B) \vee (A \otimes B) = A \oplus B$.

*Proof.* In the following, we shall prove $(i)$, and $(ii) - (iv)$ can be proved analogously.
$(i)$ Let $(A \wedge B) \oplus (A \vee B)$

$$= \Bigg[\sqrt{\min\left(\zeta_A^2(\hat{x}), \zeta_B^2(\hat{x})\right) + \max\left(\zeta_A^2(\hat{x}), \zeta_B^2(\hat{x})\right) - \min\left(\zeta_A^2(\hat{x}), \zeta_B^2(\hat{x})\right).\max\left(\zeta_A^2(\hat{x}), \zeta_B^2(\hat{x})\right)},$$
$$\max\left(\eta_A(\breve{x}), \eta_B(\breve{x})\right).\min\left(\eta_A(\breve{x}), \eta_B(\breve{x})\right), \quad \max\left(\delta_A(\hat{x}), \delta_B(\hat{x})\right).\min\left(\delta_A(\hat{x}), \delta_B(\hat{x})\right)\Bigg]$$
$$= \Big\{\Big(x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x})\Big) | x \in X\Big\}$$
$$= A \oplus B.$$ □

In the following theorems, the operator complement obey th De Morgan's laws for the operation $\oplus, \otimes, \vee, \wedge$.

**Theorem 3.8.** *If $A, B \in SNS(X)$, then*
$(i)$ $(A \oplus B)^C = A^C \otimes B^C$,
$(ii)$ $(A \otimes B)^C = A^C \oplus B^C$,
$(iii)$ $(A \oplus B)^C \leq A^C \oplus B^C$,
$(iv)$ $(A \otimes B)^C \geq A^C \otimes B^C$.

*Proof.* We shall prove $(iii), (iv)$, and $(i), (ii)$ are straightfforward.
$(iii)$ Let $(A \oplus B)^C, \forall x \in X$.

$$= \Big(\delta_A(\hat{x})\delta_B(\hat{x}), \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}\Big).$$
$$A^C \oplus B^C = \Big\{\Big(x, \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \zeta_A(\hat{x})\zeta_B(\hat{x})\Big) | x \in X\Big\}.$$

Since $\delta_A(\hat{x})\delta_B(\hat{x}) \leq \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}$

$\sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})} \geq \eta_A(\check{x})\eta_B(\check{x})$

$\sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})} \geq \zeta_A(\hat{x})\zeta_B(\hat{x})$

Hence $(A \oplus B)^C \leq A^C \oplus B^C$.

$(iv)$ Let $(A \otimes B)^C = \left(x, \sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})}, \eta_A(\check{x})\eta_B(\check{x}), \zeta_A(\hat{x})\zeta_B(\hat{x})\right)$.

$A^C \otimes B^C$

$= \left(\delta_A(\hat{x})\delta_B(\hat{x}), \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}\right)$.

Since $\sqrt{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x}) - \delta_A^2(\hat{x})\delta_B^2(\hat{x})} \geq \delta_A(\hat{x})\delta_B(\hat{x})$

$\eta_A(\check{x})\eta_B(\check{x}) \leq \sqrt{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x}) - \eta_A^2(\hat{x})\eta_B^2(\hat{x})}$

$\zeta_A(\hat{x})\zeta_B(\hat{x}) \leq \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}$

Hence $(A \otimes B)^C \geq A^C \otimes B^C$. □

**Theorem 3.9.** *If $A, B \in SNS(X)$, then*
$(i)$ $(A^C)^C = A$,
$(ii)$ $(A \vee B)^C = A^C \wedge B^C$,
$(iii)$ $(A \wedge B)^C = A^C \vee B^C$.

*Proof.* We shall prove $(ii)$ only, $(i)$ is obvious.

$A \vee B = \{(x, \max(\zeta_A(\hat{x}), \zeta_B(\hat{x})), \min(\eta_A(\check{x}), \eta_B(\check{x})), \min(\delta_A(\hat{x}), \delta_B(\hat{x})))\,|x \in X\}$

$(A \vee B)^C = \{(x, \min(\delta_A(\hat{x}), \delta_B(\hat{x})), \min(\eta_A(\check{x}), \eta_B(\check{x})), \max(\zeta_A(\hat{x}), \zeta_B(\hat{x})))\,|x \in X\}$

$\Rightarrow A^C = \{(x, \delta_A(\hat{x}), \eta_A(\check{x}), \zeta_A(\hat{x}))\,|x \in X\}$

$B^C = \{(x, \delta_B(\hat{x}), \eta_B(\check{x}), \zeta_B(\hat{x}))\,|x \in X\}$

$\Rightarrow A^C \wedge B^C = \{(x, \min(\delta_A(\hat{x}), \delta_B(\hat{x})), \min(\eta_A(\check{x}), \eta_B(\check{x})), \max(\zeta_A(\hat{x}), \zeta_B(\hat{x})))\,|x \in X\}$

Hence $(A \vee B)^C = A^C \wedge B^C$,

Similarly, we can prove that $(iii)(A \wedge B)^C = A^C \vee B^C$. □

Based on the Definition 3.1, 3.2 and 3.3., we shall next prove the algebraic properties of spherical neutrosophic sets under the operations of scalar multiplication and exponentiation.

**Theorem 3.10.** *For $A, B \in SNS(X)$ and $n, n_1, n_2 > 0$, we have*
$(i)$ $n(A \oplus B) = nA \oplus nB$,
$(ii)$ $n_1A \oplus n_2A = (n_1 + n_2)A$,
$(iii)$ $(A \otimes B)^n = A^n \otimes B^n$,
$(iv)$ $A_1^n \otimes A_2^n = A^{(n_1+n_2)}$.

*Proof.* For the two SNSs $A$ and $B$, and $n, n_1, n_2 > 0$, according to definition, we can obtain

(i) Let $n(A \oplus B)$

$= n\left\{\left(x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\check{x})\eta_B(\check{x}), \delta_A(\hat{x})\delta_B(\hat{x})\right)|x \in X\right\}$

$= \left\{\left(x, \sqrt{1 - [1 - \zeta_A^2(\hat{x})]^n[1 - \zeta_A^2(\hat{x})]^n}, [\eta_A(\check{x})\eta_B(\check{x})]^n, [\delta_A(\hat{x})\delta_B(\hat{x})]^n\right)|x \in X\right\}$

$= \left\{\left(x, \sqrt{1 - [1 - \zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})]^n}, [\eta_A(\check{x})\eta_B(\check{x})]^n, [\delta_A(\hat{x})\delta_B(\hat{x})]^n\right)|x \in X\right\}$

$nA \oplus nB$

$= \left(\sqrt{1 - [1 - \zeta_A^2(\hat{x})]^n}, [\eta_A(\check{x})]^n, [\delta_A(\hat{x})]^n\right) \oplus \left(\sqrt{1 - [1 - \zeta_B^2(\hat{x})]^n}, [\eta_B(\check{x})]^n, [\delta_B(\hat{x})]^n\right)$

$= \Big[x, \sqrt{\big(1-[1-\zeta_A^2(\hat{x})]^n + 1-[1-\zeta_A^2(\hat{x})]^n\big) - \big(1-[1-\zeta_A^2(\hat{x})]^n\big)\big(1-[1-\zeta_B^2(\hat{x})]^n\big)},$
$[\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\delta_A(\hat{x})\delta_B(\hat{x})]^n | x \in X\Big]$
$= \Big\{\Big(x, \sqrt{1-[1-\zeta_A^2(\hat{x})]^n[1-\zeta_B^2(\hat{x})]^n}, [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\delta_A(\hat{x})\delta_B(\hat{x})]^n\Big) | x \in X\Big\}$
$= \Big\{\Big(x, \sqrt{1-[1-\zeta_A^2(\hat{x})+\zeta_B^2(\hat{x})-\zeta_A^2(\hat{x})\zeta_B^2(\hat{x})]^n}, [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\delta_A(\hat{x})\delta_B(\hat{x})]^n\Big) | x \in X\Big\}$
$= n(A \oplus B)$.
(ii)Let $n_1 A \oplus n_2 B$
$= \Big[\Big(\sqrt{1-[1-\zeta_A^2(\hat{x})]^{n_1}}, [\eta_A(\breve{x})]^{n_1}, [\delta_A(\hat{x})]^{n_1}\Big) \oplus \Big(\sqrt{1-[1-\zeta_A^2(\hat{x})]^{n_2}}, [\eta_A(\breve{x})]^{n_2}, [\delta_A(\hat{x})]^{n_2}\Big)\Big]$
$= \Big[x, \sqrt{1-[1-\zeta_A^2(\hat{x})]^{n_1} + 1-[1-\zeta_A^2(\hat{x})]^{n_2} - \big(1-[1-\zeta_A^2(\hat{x})]^{n_1}\big)\big(1-[1-\zeta_A^2(\hat{x})]^{n_2}\big)},$
$[\eta_A(\breve{x})]^{n_1}[\eta_A(\breve{x})]^{n_2}, [\delta_A(\hat{x})]^{n_1}[\delta_A(\hat{x})]^{n_2} | x \in X\Big]$
$= \Big\{\Big(x, \sqrt{1-[1-\zeta_A^2(\hat{x})]^{n_1+n_2}}, [\eta_A(\breve{x})]^{n_1+n_2}, [\delta_A(\hat{x})]^{n_1+n_2}\Big) | x \in X\Big\}$
$= (n_1+n_2)A$.
(iii) Let $(A \otimes B)^n$
$= \Big[x, (\zeta_A(\hat{x})\zeta_B(\hat{x}))^n,$
$\sqrt{1-[1-\eta_A^2(\hat{x})+\eta_B^2(\hat{x})-\eta_A^2(\hat{x})\eta_B^2(\hat{x})]^n}, \sqrt{1-[1-\delta_A^2(\hat{x})+\delta_B^2(\hat{x})-\delta_A^2(\hat{x})\delta_B^2(\hat{x})]^n} | x \in X\Big]$
$= \Big[x, (\zeta_A(\hat{x})\zeta_B(\hat{x}))^n,$
$\sqrt{1-[1-\eta_A^2(\hat{x})]^n[1-\eta_B^2(\hat{x})]^n, 1-[1-\delta_A^2(\hat{x})]^n[1-\delta_B^2(\hat{x})]^n} | x \in X\Big]$
$A^n \otimes B^n$
$= \Big[x, (\zeta_A(\hat{x})\zeta_B(\hat{x}))^n,$
$\sqrt{1-[1-\eta_A^2(\hat{x})]^n + 1-[1-\eta_B^2(\hat{x})]^n - \big(1-[1-\eta_A^2(\hat{x})]^n\big)\big(1-[1-\eta_B^2(\hat{x})]^n\big)},$
$\sqrt{1-[1-\delta_A^2(\hat{x})]^n + 1-[1-\delta_B^2(\hat{x})]^n - \big(1-[1-\delta_A^2(\hat{x})]^n\big)\big(1-[1-\delta_B^2(\hat{x})]^n\big)} | x \in X\Big]$
$= \Big((\zeta_A(\hat{x})\zeta_B(\hat{x}))^n, \sqrt{1-[1-\eta_A^2(\hat{x})]^n[1-\eta_B^2(\hat{x})]^n}, \sqrt{1-[1-\delta_A^2(\hat{x})]^n[1-\delta_B^2(\hat{x})]^n}\Big)$
$= (A \otimes B)^n$.
(iv) Let $A^{n_1} \otimes A^{n_2}$

$$= \left[ \begin{array}{l} (\zeta_A(\hat{x}))^{n_1+n_2}, \\ \sqrt{1-[1-\eta_A^2(\hat{x})]^{n_1} + 1-[1-\eta_A^2(\hat{x})]^{n_2} - \big(1-[1-\eta_A^2(\hat{x})]^{n_1}\big)\big(1-[1-\eta_A^2(\hat{x})]^{n-2}\big)}, \\ \sqrt{1-[1-\delta_A^2(\hat{x})]^{n_1} + 1-[1-\delta_A^2(\hat{x})]^{n-2} - \big(1-[1-\delta_A^2(\hat{x})]^{n_1}\big)\big(1-[1-\delta_A^2(\hat{x})]^{n_2}\big)} \end{array} \right]$$

$= \Big\{\Big(x, (\zeta_A(\hat{x}))^{n_1+n_2}, \sqrt{1-[1-\eta_A^2(\hat{x})]^{n_1+n_2}}, \sqrt{1-[1-\delta_A^2(\hat{x})]^{n_1+n_2}}\Big) | x \in X\Big\}$
$= A^{(n_1+n_2)}$.
Hence proved. □

**Theorem 3.11.** *For $A, B \in SNS(X)$ and $n > 0$, we have*
*(i) $nA \leq nB$,*
*(ii) $A^n \leq B^n$.*

*Proof.* (*i*) Let $A \leq B$
$\Rightarrow \zeta_A(\hat{x}) \leq \zeta_B(\hat{x})$ and $\eta_A(\breve{x}) \geq \eta_B(\breve{x})$ and $\delta_A(\hat{x}) \geq \delta_B(\hat{x})$ for all $x \in X$.

$\Rightarrow \sqrt{1-[1-\zeta_A^2(\hat{x})]^n} \leq \sqrt{1-[1-\zeta_B^2(\hat{x})]^n}$,
$[\eta_A(\breve{x})]^n \geq [\eta_B(\breve{x})]^n$ and
$[\delta_A(\hat{x})]^n \geq [\delta_B(\hat{x})]^n$. for all $x \in X$.
$(ii)$ Also, $[\zeta_A(\hat{x})]^n \geq [\zeta_B(\hat{x})]^n$,
$\sqrt{1-[1-\eta_A^2(\hat{x})]^n} \leq \sqrt{1-[1-\eta_B^2(\hat{x})]^n}$,
$\sqrt{1-[1-\delta_A^2(\hat{x})]^n} \leq \sqrt{1-[1-\delta_B^2(\hat{x})]^n}$, for all $x \in X$. □

**Theorem 3.12.** *For $A, B \in SNS(X)$ and $n > 0$, we have*
*$(i)$ $n(A \wedge B) = nA \wedge nB$,*
*$(ii)$ $n(A \vee B) = nA \vee nB$.*

*Proof.* $(i)$ Let $n(A \wedge B)$

$$= \Big[x, \sqrt{1-[1-\min\left(\zeta_A^2(\hat{x}), \zeta_B^2(\hat{x})\right)]^n}, \max\left([\eta_A(\breve{x})]^n, [\eta_B(\breve{x})]^n\right), \max\left([\delta_A(\hat{x})]^n, [\delta_B(\hat{x})]^n\right) | x \in X\Big]$$

$$= \Big[x, \sqrt{1-[\max\left(1-\zeta_A^2(\hat{x}), 1-\zeta_B^2(\hat{x})\right)]^n}, \max\left([\eta_A(\breve{x})]^n, [\eta_B(\breve{x})]^n\right), \max\left([\delta_A(\hat{x})]^n, [\delta_B(\hat{x})]^n\right) | x \in X\Big]$$

$$= \Big[x, \sqrt{1-\left(\max\left([1-\zeta_A^2(\hat{x})]^n, [1-\zeta_B^2(\hat{x})]^n\right)\right)}, \max\left([\eta_A(\breve{x})]^n, [\eta_B(\breve{x})]^n\right), \max\left([\delta_A(\hat{x})]^n, [\delta_B(\hat{x})]^n\right) | x \in X\Big]$$

$$= \left[\max\left(\sqrt{1-[1-\zeta_A^2(\hat{x})]^n}, \sqrt{1-[1-\zeta_B^2(\hat{x})]^n}\right), \max\left([\eta_A(\breve{x})]^n, [\eta_B(\breve{x})]^n\right), \max\left([\delta_A(\hat{x})]^n, [\delta_B(\hat{x})]^n\right)\right]$$

$$= nA \wedge nB.$$

Hence $n(A \wedge B) = nA \wedge nB$,
Similarly, we can prove that $(ii) n(A \vee B) = nA \vee nB$. □

**Theorem 3.13.** *For $A, B \in SNS(X)$ and $n > 0$, we have*
*$(i)$ $(A \wedge B)^n = A^n \wedge B^n$,*
*$(ii)$ $(A \vee B)^n = A^n \vee B^n$.*

*Proof.* $(i)$ Let $(A \wedge B)^n$

$$= \left[x, \min\left([\zeta_A(\hat{x})]^n, [\zeta_B(\hat{x})]^n\right), \sqrt{1-[\max\left(1-\eta_A^2(\hat{x}), 1-\eta_B^2(\hat{x})\right)]^n}, \sqrt{1-[\max\left(1-\delta_A^2(\hat{x}), 1-\delta_B^2(\hat{x})\right)]^n} | x \in X\right]$$

$$= \Big[x, \min\left([\zeta_A(\hat{x})]^n, [\zeta_B(\hat{x})]^n\right), \sqrt{1-\left(\min\left([1-\eta_A^2(\hat{x})]^n, [1-\eta_B^2(\hat{x})]^n\right)\right)}, \sqrt{1-\left(\min\left([1-\delta_A^2(\hat{x})]^n, [1-\delta_B^2(\hat{x})]^n\right)\right)} | x \in X\Big]$$

$$= \Big[x, \min\left([\zeta_A(\hat{x})]^n, [\zeta_B(\hat{x})]^n\right), \max\left(\sqrt{1-[1-\eta_A^2(\hat{x})]^n}, \sqrt{1-[1-\eta_B^2(\hat{x})]^n}\right), \max\left(\sqrt{1-[1-\delta_A^2(\hat{x})]^n}, \sqrt{1-[1-\delta_B^2(\hat{x})]^n}\right) | x \in X\Big]$$

$A^n \wedge B^n$

$$= \left[ \left( [\zeta_A(\hat{x})]^n, \sqrt{1-[1-\eta_A^2(\hat{x})]^n}, \sqrt{1-[1-\delta_A^2(\hat{x})]^n} \right) \bigwedge \right.$$
$$\left. \left( [\zeta_B(\hat{x})]^n, \sqrt{1-[1-\eta_B^2(\hat{x})]^n}, \sqrt{1-[1-\delta_B^2(\hat{x})]^n} \right) \right]$$
$$= \left[ x, \min\left([\zeta_A(\hat{x})]^n, [\zeta_B(\hat{x})]^n\right), \max\left( \sqrt{1-[1-\eta_A^2(\hat{x})]^n}, \sqrt{1-[1-\eta_B^2(\hat{x})]^n} \right), \right.$$
$$\left. \max\left( \sqrt{1-[1-\delta_A^2(\hat{x})]^n}, \sqrt{1-[1-\delta_B^2(\hat{x})]^n} \right) \right]$$
$$= (A \wedge B)^n.$$

Hence $(A \wedge B)^n = A^n \wedge B^n$,
Similarly, we can prove that $(ii)(A \vee B)^n = A^n \vee B^n$. □

**Theorem 3.14.** *For $A, B \in SNS(X)$ and $n > 0$, we have*
$(A \oplus B)^n \neq A^n \oplus B^n$.

*Proof.* Let $(A \oplus B)^n$

$$= \Big[ x, \left( \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})} \right)^n, \sqrt{1-[1-\eta_A^2(\hat{x})\eta_B^2(\hat{x})]^n},$$
$$\sqrt{1-[1-\delta_A^2(\hat{x})\delta_B^2(\hat{x})]^n} | x \in X | x \in X \Big]$$

$$A^n = \left\{ \left( x, [\zeta_A(\hat{x})]^n, \sqrt{1-[1-\eta_A^2(\hat{x})]^n}, \sqrt{1-[1-\delta_A^2(\hat{x})]^n} \right) | x \in X \right\}$$
$$B^n = \left\{ \left( x, [\zeta_B(\hat{x})]^n, \sqrt{1-[1-\eta_B^2(\hat{x})]^n}, \sqrt{1-[1-\delta_B^2(\hat{x})]^n} \right) | x \in X \right\}$$

$A^n \oplus B^n$

$$= \Big[ x, \sqrt{[\zeta_A^n(\hat{x})]^2 + [\zeta_B^n(\hat{x})]^2 - [\zeta_A^n(\hat{x})]^2[\zeta_B^n(\hat{x})]^2}, \left( \sqrt{1-[1-\eta_A^2(\hat{x})]^n} \right)^n .$$
$$\left( \sqrt{1-[1-\eta_B^2(\hat{x})]^n} \right)^n, \left( \sqrt{1-[1-\delta_A^2(\hat{x})]^n} \right)^n . \left( \sqrt{1-[1-\delta_A^2(\hat{x})]^n} \right)^n | x \in X \Big]$$

Hence $(A \oplus B)^n \neq A^n \oplus B^n$. □

## 4. New operation (@) on spherical neutrosophic sets

In this section, we define a new operation(@) on spherical neutrosophic sets and prove their desirable properties.

**Definition 4.1.** *If $A$ and $B$ are two Spherical neutrosophic sets, then*

$$A@B = \left\{ \left( x, \sqrt{\frac{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x})}{2}}, \sqrt{\frac{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x})}{2}}, \sqrt{\frac{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x})}{2}} \right) | x \in X \right\}.$$

**Remark 4.1.** *Obviously, for every two spherical neutrosophic sets $A$ and $B$, then $A@B$ is a spherical neutrosophic set.*

*Simple illustration given: For $A@B$,*

$$0 \leq \frac{\zeta_A(\hat{x}) + \zeta_B(\hat{x})}{2} + \tfrac{\eta_A(\breve{x}) + \eta_B(\breve{x})}{2} + \frac{\delta_A(\hat{x}) + \delta_B(\hat{x})}{2}$$
$$\leq \frac{\zeta_A(\hat{x}) + \eta_A(\breve{x}) + \delta_A(\hat{x})}{2} + \frac{\zeta_B(\hat{x}) + \eta_B(\breve{x}) + \delta_B(\hat{x})}{2} \leq \frac{3}{2} + \frac{3}{2} = 3.$$

**Theorem 4.1.** *For any spherical neutrosophic set $A$, $A@A = A$.*

*Proof.* $A@A$

$$= \left\{ \left( x, \sqrt{\frac{\zeta_A^2(\hat{x}) + \zeta_A^2(\hat{x})}{2}}, \sqrt{\frac{\eta_A^2(\hat{x}) + \eta_A^2(\hat{x})}{2}}, \sqrt{\frac{\delta_A^2(\hat{x}) + \delta_A^2(\hat{x})}{2}} \right) |x \in X \right\}$$

$$= \left\{ \left( x, \left( \sqrt{\frac{\zeta_A^2(\hat{x}) + \zeta_A^2(\hat{x})}{2}} \right)^2, \left( \sqrt{\frac{\eta_A^2(\hat{x}) + \eta_A^2(\hat{x})}{2}} \right)^2, \left( \sqrt{\frac{\delta_A^2(\hat{x}) + \delta_A^2(\hat{x})}{2}} \right)^2 \right) |x \in X \right\}$$

$$= \left\{ \left( x, \frac{2\zeta_A^2(\hat{x})}{2}, \frac{2\eta_A^2(\hat{x})}{2}, \frac{2\delta_A^2(\hat{x})}{2} \right) |x \in X \right\}$$

$= \{(x, \zeta_A(\hat{x}), \eta_A(\breve{x}), \delta_A(\hat{x}))\, |x \in X\}$. Since $\zeta_A^2(\hat{x}) \leq \zeta_A(\hat{x}), \eta_A^2(\hat{x}) \leq \eta_A(\breve{x}), \delta_A^2(\hat{x}) \leq \delta_A(\hat{x})$
$= A$. □

**Remark 4.2.** *If $a, b \in [0,1]$, then $ab \leq \dfrac{a+b}{2}, \dfrac{a+b}{2} \leq a + b - ab$.*

**Theorem 4.2.** *If $A, B \in SNS(X)$, then*
*(i)* $(A \oplus B) \vee (A@B) = A \oplus B$,
*(ii)* $(A \otimes B) \wedge (A@B) = A \otimes B$,
*(iii)* $(A \oplus B) \wedge (A@B) = A@B$,
*(iv)* $(A \otimes B) \vee (A@B) = A@B$.

*Proof.* we shall prove (i) and (iii), (ii) and (iv) can be proved analogously.
*(i)* Let $(A \oplus B) \vee (A@B)$

$$= \left[ \max \left( \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \sqrt{\frac{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x})}{2}} \right), \right.$$
$$\left. \min \left( \eta_A(\breve{x})\eta_B(\breve{x}), \sqrt{\frac{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x})}{2}} \right), \min \left( \delta_A(\hat{x})\delta_B(\hat{x}), \sqrt{\frac{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x})}{2}} \right) \right]$$

$$= \left\{ \left( x, \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \eta_A(\breve{x})\eta_B(\breve{x}), \delta_A(\hat{x})\delta_B(\hat{x}) \right) |x \in X \right\}$$

$= A \oplus B$.
*(iii)* $(A \oplus B) \wedge (A@B)$

$$= \left[ \min \left( \sqrt{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x}) - \zeta_A^2(\hat{x})\zeta_B^2(\hat{x})}, \sqrt{\frac{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x})}{2}} \right), \right.$$
$$\left. \max \left( \eta_A(\breve{x})\eta_B(\breve{x}), \sqrt{\frac{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x})}{2}} \right), \max \left( \delta_A(\hat{x})\delta_B(\hat{x}), \sqrt{\frac{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x})}{2}} \right) \right]$$

$$= \left\{ \left( x, \sqrt{\frac{\zeta_A^2(\hat{x}) + \zeta_B^2(\hat{x})}{2}}, \sqrt{\frac{\eta_A^2(\hat{x}) + \eta_B^2(\hat{x})}{2}}, \sqrt{\frac{\delta_A^2(\hat{x}) + \delta_B^2(\hat{x})}{2}} \right) |x \in X \right\}$$

$= A@B$,
Hence proved. □

## 5. Conclusion

In this paper, Algebraic sum and Algebraic product of spherical neutrosophic sets are defined. Then some properties such as, idempotency, commutativity, associativity, absorption law, distributivity and De Morgan's laws over complement are proved. Further, scalar multiplication ($nA$) and exponentiation ($A^n$) operations of a spherical neutrosophic sets are constructed. Finally, defined a new operation(@) on spherical neutrosophic sets and discussed distributive laws. In the future, the application of the proposed aggregating operators of SNSs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

**Acknowledgement.** The author is thankful to the Editor-in-chief for the technical comments, and to the anonymous reviewers for their suggestions, which have improved the quality of the paper.

## References

[1] Atanassov, K., (1986), Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20, pp. 87-96.
[2] Cuong, B.C., (2001), Picture fuzzy sets, journal of computer science and cybernetics, 30(4), pp. 409-420.
[3] Das, S., Roy, B. K. and Kar, M. B., (2020), Neutrosophic fuzzy set and its application in decision making, J Ambient Intell Human Comput., 11, pp. 5017-5029.
[4] Dogra, D. and Pal, M., (2020), m-polar picture fuzzy ideal of a BCK algebra, International Journal of Computational Intelligence Systems, 13(1), pp. 409-420.
[5] Dogra, D. and Pal, M., (2020), Picture fuzzy matrix and its application, Soft Computing, 24, pp. 9413-9428.
[6] Dogra, D. and Pal, M., (2021), Picture fuzzy subring of a crisp ring, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 91, pp. 429–434.
[7] Dogra, D., Pal, M. and Xin, Q., (2022), Picture fuzzy sub-hyperspace of a hyper vector space and its application in decision making problem , AIMS Mathematics, 7(7), pp. 13361-13382.
[8] Dogra, D. and Pal, M., (2023), Picture fuzzy subspace of a crisp vector space, Kragujevac Journal of Mathematics, 47(4), pp. 577–597.
[9] Dogra, D. and Pal, M., (2023), Picture fuzzy subgroup, Kragujevac Journal of Mathematics, 47(6), Pp. 911–933.
[10] Garg, H. and Nancy, (2016a), Single-valued neutrosophic Entropy of order alpha, Neutrosophic Sets Syst, 14, pp. 21-28.
[11] Garg, H., (2017), Some Picture fuzzy aggregation operators and their applications to multicriteria decision-making, Arabian Journal for Science and Engineering, 42(12), pp. 5275-5290.
[12] Gündogdu, F. K. and Kahraman, C., (2018), Spherical fuzzy sets and spherical fuzzy TOPSIS method, J. Intell. Fuzzy Syst., 36, pp. 1-16.
[13] Kandasamy, W. B. V. and Smarandache, F., (2004), Fuzzy Relational Maps and Neutrosophic Relational Maps, HEXIS Church Rock, book, 302 pages.
[14] Silambarasan, I. and Sriram, S., (2019), Implication operator on Pythagorean Fuzzy Set, International Journal of Scientific & Technology Research, 8 (8), pp. 1505-1509.
[15] Silambarasan, I., (2020), New operators for Fermatean fuzzy sets, Annals of Communucations in Mathematics, 3(2), pp. 116-131.
[16] Silambarasan, I., (2021), Some Algebraic properties of Picture fuzzy sets, Bull. Int. Math. Virtual Inst., 11(3), pp. 429-442.
[17] Smarandache, F., (1998), Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. Rehoboth: American Research Press.
[18] Smarandache, F., (1999), A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press.
[19] Smarandache, F., (2002), A unifying field in logics: neutrosophic logics. Multiple Valued Logic, 8(3), pp. 385-438.
[20] Smarandache, F., (2005), Neutrosophic set, a generalization of intuitionistic fuzzy sets, International Journal of Pure and Applied Mathematics, 24, pp. 287-297.
[21] Smarandache, F., (2006), Neutrosophic set- a generalization of intuitionistic fuzzy set, Granular Computing, 2006 IEEE, International Conference, pp. 38-42. doi:10.1109/GRC.2006.1635754.
[22] Smarandache, F., (2010), Neutrosophic set-a generalization of intuitionistic fuzzy set, Journal of Defence Resources Management, 1(1), pp. 107-116.
[23] Smarandache, F.,(2019), Neutrosophic Set as Generalization of Intuitioonistic Fuzzy Set, Picture Fuzzy Set and Spherical Fuzzy Set, and its Physical Applications, Joint Fall Meeting of the Texas Sections of American Physical Society (APS), AAPT and Zone 13 of the SPS, FridaySaturday, October 25–26; Lubbock, Texas, USA,

[24] Wang, H., Smarandache, F., Zhang, Y. Q. and Sunderraman, R., (2010), Single valued neutrosophic sets, Multispace Multistruct, 4, pp. 410-413.
[25] Yager, R. R., Pythagorean fuzzy subsets, In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, 2013, pp. 57-61.
[26] Yager, R. R. and Abbasov, A. M., (2014), Pythagorean membership grades, complex numbers, and decision making, International Journal of Intelligent System, 28, pp. 436-452.
[27] Zadeh, L. A., (1965), Fuzzy sets, Information and Control, 8(3), pp. 338-356.
[28] Zhang, X. L. and Xu, Z. S., (2014), Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, International Journal of Intelligent Systems, 29, pp. 1061-1078.

**Dr. A. Manimaran** received his Ph.D. degree from VIT University, Vellore, Tamilnadu, India. He is an Assistant professor in VIT University, Vellore.

**Dr. V. Muthukumaran** received his Ph.D. degree in Mathematics from the School of Advanced Sciences, Vellore Institute of Technology, Vellore in 2019. At present, he is working as an Assistant Professor in the Department of Mathematics, SRM Institute of Science and Technology in Kattankulathur. His current research interests include Algebraic cryptography, Fuzzy Image Processing, Machine learning, and Data mining.

**Dr. R. Vijaykrishnaraj** received his Ph.D. degree from Annamalai University, Chidambaram, Tamilnadu, India. He is an Assistant Professor in Bharath Institute of Higher Education and Research.

**Dr. I. Silambarasan** for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.2.