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Thermo-microstretch elastic bodies and plane waves

E Inan[1] and A Kiris[2]

[1] Faculty of Arts and Sciences, Işık University, 34980, Şile, İstanbul, Turkey
[2] Faculty of Science and Letters, İstanbul Technical Univ., 34469, Maslak, İstanbul, Turkey

E-mail: inan@isikun.edu.tr, kiris@itu.edu.tr

Abstract. In the present work, vibration problems of rectangular plates are considered for the determination of upper bounds to the unknown microstretch material properties. The frequencies are obtained by extending the Ritz method to this case. The analysis shows that some additional frequencies characterizing the microstretch effects appear among the classical frequencies. Furthermore, by the increasing values of the microstretch constants, the additional frequencies disappear and only the classical frequencies remain in the spectrum. Considering this phenomenon, an optimization problem is established for the identification of the upper bounds of microstretch elastic constants. In the second part of the work, thermal effects are considered and several theories are discussed. Finally, propagation of the plane waves is investigated.

1. Introduction

It is well known that the linear theory of elasticity is unable to explain the behaviour of the materials having complex microstructure such as polymers, porous media and micro damaged materials, etc. Modelling such materials with Eringen's [1] microstretch theory which assumes that every particles of the material may do both microrotation and volumetric microelongation in addition to the bulk deformation is more convenient. Success of this theory depends on the correctness of the choice of system parameters. Dependence of the dynamic response of the medium to material properties allows us to establish a vibration analysis [2] for the determination of such material constants similar to the classical cases [3–11]. This analysis is based on 3-D microstretch theory and extended Ritz method. Following Zhou et al. [12], the triplicate Chebyshev polynomial series are used to describe plate deflections. The wave propagation problems in micropolar and microstretch media are discussed in [13, 14], and it is shown that two and three new non-classical waves appear in micropolar and microstretch cases, respectively. Parallel to these results, we found some additional frequencies due to the microstretch character, among the classical frequencies [2]. As it is expected, these additional frequencies disappear when the microstretch material constants are taken zero. Besides, we observed that these additional frequencies are more sensitive to the variation of micro elastic constants than the classical frequencies. Therefore, the values of additional frequencies rapidly increase by the increase of micro constants and then considerable amount of them move out among classical frequencies. During the variation of microstretch constants, the values of classical frequencies remain same up to some critical values, but they also begin to deviate after these critical values. This phenomenon tells us that the microstructure becomes more dominant and starts to affect the macro properties. For instance, considering this model as representing a damaged body, we may conclude that the development and the growth of micro cracking start to affect the macro properties of the body. So, we may define these threshold values as the upper bounds for the microstretch material parameters. That means, the material is not a microstretch body anymore. Beyond these threshold values of the micro elastic constants, the material loses its microstretch character. So, this is the key point of this study that will be used to construct an optimization problem to determine the upper bounds of the microstretch properties. To find more specific values for the micro constants, we need to know some experimental data for the frequencies of the material. But, it needs more sophisticated equipments and experimental techniques that are not available at the moment. Thus, we may use experimental results due to classical cases which give us only the upper bounds for micro constants.

Another important problem of damage mechanics is to include thermal effects, and thermo-microstretch theory can be used. In this case, some additional microthermal constants will occur. To determine upper bounds for the new constants, same method may not work properly due to the special form of the thermal analysis and some modifications should be done in the method. Then, it is left to a further study. Thus, we give here an analysis of the comparison of several thermal theories and then investigate the plane wave propagation in a thermo-microstretch material.

2. Fundamental equations and wave propagation in microstretch medium

The constitutive equations for a linear homogeneous, isotropic, microstretch elastic solid may be given as follows [15]

$$\begin{aligned} t_{kl} &= \lambda\, \varepsilon_{mm}\, \delta_{kl} + (\mu+\kappa)\, \varepsilon_{kl} + \mu\, \varepsilon_{lk} + \lambda_0\, \theta\, \delta_{kl}, \\ m_{kl} &= \alpha\, \gamma_{mm}\, \delta_{kl} + \beta\, \gamma_{kl} + \gamma\, \gamma_{lk}, \\ m_k &= a_0\, \theta_{,k}, \\ s - t &= \lambda_1\, \theta + \lambda_0\, \varepsilon_{kk}. \end{aligned} \tag{1}$$

Here, $t_{kl},\ m_{kl}$ are the stress and couple stress tensors, m_k is the microstretch vector and $s = s_{kk}$, $t = t_{kk}$. Strain tensors of a microstretch medium are

$$\varepsilon_{kl} = u_{l,k} + e_{lkm}\phi_m, \quad \gamma_{kl} = \phi_{k,l}, \quad \gamma_k = \theta_{,k}. \tag{2}$$

Here, $\lambda,\ \mu$ are Lamé constant and shear modulus, $\kappa,\ \alpha,\ \beta,\ \gamma$ are micropolar constants, $\lambda_0,\ \lambda_1$ and a_0 are microstretch constants, ρ is the mass density, j is micro-inertia, and $\boldsymbol{u},\ \boldsymbol{\phi}$ and θ are displacement and microrotation vectors and microstretch scalar, respectively.

Accordingly, the equations of motion in a linear homogeneous and isotropic microstretch elastic solid are given as

$$\begin{aligned} &(c_1^2 + c_3^2)\nabla\nabla\cdot\boldsymbol{u} - (c_2^2 + c_3^2)\nabla\times\nabla\times\boldsymbol{u} + c_3^2\nabla\times\boldsymbol{\phi} + \bar{\lambda}_0\boldsymbol{\nabla}\theta = \ddot{\boldsymbol{u}} \\ &(c_4^2 + c_5^2)\nabla\nabla\cdot\boldsymbol{\phi} - c_4^2\nabla\times\nabla\times\boldsymbol{\phi} + \omega_0^2\nabla\times\boldsymbol{u} - 2\omega_0^2\boldsymbol{\phi} = \ddot{\boldsymbol{\phi}} \\ &c_6^2\Delta\theta - c_7^2\theta - c_8^2\nabla\cdot\boldsymbol{u} = \ddot{\theta} \end{aligned} \tag{3}$$

where

$$\begin{aligned} &c_1^2 = \tfrac{(\lambda+2\mu)}{\rho} \quad c_2^2 = \tfrac{\mu}{\rho}, \quad c_3^2 = \tfrac{\kappa}{\rho}, \quad c_4^2 = \tfrac{\gamma}{\rho j}, \quad c_5^2 = \tfrac{(\alpha+\beta)}{\rho j}, \\ &c_6^2 = \tfrac{2a_0}{\rho j}, \quad c_7^2 = \tfrac{2\lambda_1}{3\rho j}, \quad c_8^2 = \tfrac{2\lambda_0}{3\rho j}, \quad \omega_0^2 = \tfrac{c_3^2}{j}, \quad \bar{\lambda}_0 = \tfrac{\lambda_0}{\rho}. \end{aligned} \tag{4}$$

The displacement and microrotation vectors can be decomposed to their solenoidal and rotational parts by using Helmholtz decomposition as

$$\begin{aligned} \boldsymbol{u} &= \nabla\bar{u} + \nabla\times\boldsymbol{U}, \quad \nabla\cdot\boldsymbol{U} = 0, \\ \boldsymbol{\phi} &= \nabla\bar{\boldsymbol{\phi}} + \nabla\times\boldsymbol{\Phi}, \quad \nabla\cdot\boldsymbol{\Phi} = 0. \end{aligned} \tag{5}$$

Now substituting these potentials into equations of motion, we arrive at the following equations:

$$\begin{aligned}
&(c_1^2+c_3^2)\Delta\bar{u}+\bar{\lambda}_0\theta=\ddot{\bar{u}},\\
&c_6^2\Delta\theta-c_7^2\theta-c_8^2\Delta\bar{u}=\ddot{\theta},\\
&(c_2^2+c_3^2)\Delta\boldsymbol{U}+c_3^2\nabla\times\boldsymbol{\Phi}=\ddot{\boldsymbol{U}},\\
&c_4^2\Delta\boldsymbol{\Phi}-2\omega_0^2\boldsymbol{\Phi}+\omega_0^2\nabla\times\boldsymbol{U}=\ddot{\boldsymbol{\Phi}},\\
&(c_4^2+c_5^2)\Delta\bar{\phi}-2\omega_0^2\bar{\phi}=\ddot{\bar{\phi}}.
\end{aligned} \tag{6}$$

3. 3D vibration analysis of the microstretch plate

We consider a homogeneous isotropic rectangular plate with length a, width b and thickness h [2]. A Cartesian coordinate system (x_1, x_2, x_3) is located at the mid-point of the plate. The maximum energy functional of a plate is

$$\Pi = V_{\max} - T_{\max} \tag{7}$$

where $V_{\max}$ and $T_{\max}$ are linear elastic strain energy and kinetic energy, respectively. Assuming harmonic-time dependence, displacement, microrotation and microstretch components of the microstretch plate undergoing free vibration may be written in terms of amplitude functions as follows

$$\{\boldsymbol{u}(x_1,x_2,x_3,t),\boldsymbol{\phi}(x_1,x_2,x_3,t),\theta(x_1,x_2,x_3,t)\}=\{\boldsymbol{U}(x_1,x_2,x_3),\boldsymbol{\Phi}(x_1,x_2,x_3),\Theta(x_1,x_2,x_3)\}\;e^{i\omega t} \tag{8}$$

Here, ω denotes the natural frequency. Now introducing the following non dimensional parameters;

$$\xi=\frac{2x_1}{a},\quad \eta=\frac{2x_2}{b},\quad \zeta=\frac{2x_3}{h}, \tag{9}$$

the elastic strain energy $V_{\max}$ and the kinetic energy $T_{\max}$ take the following forms:

$$\begin{aligned}
V=\tfrac{h}{4\alpha_1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}&\big[\lambda\,\bar{\Lambda}_1^2+(2\mu+\kappa)\bar{\Lambda}_2+2\mu\bar{\Lambda}_3+(\mu+\kappa)\bar{\Lambda}_4+\alpha\bar{\Lambda}_5^2+(\beta+\gamma)\bar{\Lambda}_6\\
&+2\beta\bar{\Lambda}_7+\gamma\bar{\Lambda}_8+a_0\bar{\Lambda}_9\big]\,d\xi\,d\eta\,d\zeta\\
+\tfrac{b\,h}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}&\left[\kappa\,\bar{\bar{\Lambda}}_3+2\,\lambda_0\,\Theta\,\bar{\Lambda}_1\right]d\xi\,d\eta\,d\zeta+\tfrac{a\,b\,h}{16}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left[\kappa\bar{\bar{\bar{\Lambda}}}_3+\lambda_1\Theta^2\right]d\xi\,d\eta\,d\zeta
\end{aligned} \tag{10}$$

and

$$T=\frac{\rho}{16}a\,b\,h\,\omega^2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left\{\left[U_1^2+U_2^2+U_3^2\right]+j\left[\Phi_1^2+\Phi_2^2+\Phi_3^2\right]+3\,j\,\Theta^2\right\}d\xi\,d\eta\,d\zeta \tag{11}$$

where

$$\begin{aligned}
&\bar{\Lambda}_1=\bar{\varepsilon}_{\xi\xi}+\bar{\varepsilon}_{\eta\eta}+\bar{\varepsilon}_{\zeta\zeta},\quad \bar{\Lambda}_2=\bar{\varepsilon}_{\xi\xi}^2+\bar{\varepsilon}_{\eta\eta}^2+\bar{\varepsilon}_{\zeta\zeta}^2,\\
&\bar{\Lambda}_3={}_1\bar{\varepsilon}_{\xi\eta}\,{}_1\bar{\varepsilon}_{\eta\xi}+{}_1\bar{\varepsilon}_{\xi\zeta}\,{}_1\bar{\varepsilon}_{\zeta\xi}+{}_1\bar{\varepsilon}_{\eta\zeta}\,{}_1\bar{\varepsilon}_{\zeta\eta},\\
&\bar{\bar{\Lambda}}_3=\left({}_1\bar{\varepsilon}_{\xi\eta}-{}_1\bar{\varepsilon}_{\eta\xi}\right){}_2\bar{\varepsilon}_{\xi\eta}+\left({}_1\bar{\varepsilon}_{\xi\zeta}-{}_1\bar{\varepsilon}_{\zeta\xi}\right){}_2\bar{\varepsilon}_{\xi\zeta}+\left({}_1\bar{\varepsilon}_{\eta\zeta}-{}_1\bar{\varepsilon}_{\zeta\eta}\right){}_2\bar{\varepsilon}_{\eta\zeta},\\
&\bar{\bar{\bar{\Lambda}}}_3={}_2\bar{\varepsilon}_{\xi\eta}^2+{}_2\bar{\varepsilon}_{\xi\zeta}^2+{}_2\bar{\varepsilon}_{\eta\zeta}^2,\quad \bar{\Lambda}_4={}_1\bar{\varepsilon}_{\xi\eta}^2+{}_1\bar{\varepsilon}_{\eta\xi}^2+{}_1\bar{\varepsilon}_{\xi\zeta}^2+{}_1\bar{\varepsilon}_{\zeta\xi}^2+{}_1\bar{\varepsilon}_{\eta\zeta}^2+{}_1\bar{\varepsilon}_{\zeta\eta}^2,\\
&\bar{\Lambda}_5=\bar{\gamma}_{\xi\xi}+\bar{\gamma}_{\eta\eta}+\bar{\gamma}_{\zeta\zeta},\quad \bar{\Lambda}_6=\bar{\gamma}_{\xi\xi}^2+\bar{\gamma}_{\eta\eta}^2+\bar{\gamma}_{\zeta\zeta}^2,\\
&\bar{\Lambda}_7=\bar{\gamma}_{\xi\eta}\,\bar{\gamma}_{\eta\xi}+\bar{\gamma}_{\xi\zeta}\,\bar{\gamma}_{\zeta\xi}+\bar{\gamma}_{\eta\zeta}\,\bar{\gamma}_{\zeta\eta},\quad \bar{\Lambda}_8=\bar{\gamma}_{\xi\eta}^2+\bar{\gamma}_{\eta\xi}^2+\bar{\gamma}_{\xi\zeta}^2+\bar{\gamma}_{\zeta\xi}^2+\bar{\gamma}_{\eta\zeta}^2+\bar{\gamma}_{\zeta\eta}^2,\\
&\bar{\Lambda}_9=\bar{\Theta}_{,\xi}^2+\bar{\Theta}_{,\eta}^2+\bar{\Theta}_{,\zeta}^2
\end{aligned} \tag{12}$$

and

$$\begin{array}{l}\bar{\varepsilon}_{\xi\xi}=\frac{\partial U_1}{\partial\xi},\quad \bar{\varepsilon}_{\eta\eta}=\alpha_1\frac{\partial U_2}{\partial\eta},\quad \bar{\varepsilon}_{\zeta\zeta}=\frac{\alpha_1}{\alpha_2}\frac{\partial U_3}{\partial\zeta},\quad {}_1\bar{\varepsilon}_{\xi\eta}=\frac{\partial U_2}{\partial\xi},\quad {}_1\bar{\varepsilon}_{\eta\xi}=\alpha_1\frac{\partial U_1}{\partial\eta},\quad {}_1\bar{\varepsilon}_{\xi\zeta}=\frac{\partial U_3}{\partial\xi},\\ {}_1\bar{\varepsilon}_{\zeta\xi}=\frac{\alpha_1}{\alpha_2}\frac{\partial U_1}{\partial\zeta},\quad {}_1\bar{\varepsilon}_{\eta\zeta}=\alpha_1\frac{\partial U_3}{\partial\eta},\quad {}_1\bar{\varepsilon}_{\zeta\eta}=\frac{\alpha_1}{\alpha_2}\frac{\partial U_2}{\partial\zeta},\quad {}_2\bar{\varepsilon}_{ij}=e_{jik}\Phi_k,\quad (i,j,k=\xi,\,\eta,\,\zeta),\\ \bar{\gamma}_{\xi\xi}=\frac{\partial\Phi_1}{\partial\xi},\quad \bar{\gamma}_{\eta\eta}=\alpha_1\frac{\partial\Phi_2}{\partial\eta},\quad \bar{\gamma}_{\zeta\zeta}=\frac{\alpha_1}{\alpha_2}\frac{\partial\Phi_3}{\partial\zeta},\quad \bar{\gamma}_{\xi\eta}=\alpha_1\frac{\partial\Phi_1}{\partial\eta},\quad \bar{\gamma}_{\eta\xi}=\frac{\partial\Phi_2}{\partial\xi},\quad \bar{\gamma}_{\xi\zeta}=\frac{\alpha_1}{\alpha_2}\frac{\partial\Phi_1}{\partial\zeta},\\ \bar{\gamma}_{\zeta\xi}=\frac{\partial\Phi_3}{\partial\xi},\quad \bar{\gamma}_{\eta\zeta}=\frac{\alpha_1}{\alpha_2}\frac{\partial\Phi_2}{\partial\zeta},\quad \bar{\gamma}_{\zeta\eta}=\alpha_1\frac{\partial\Phi_3}{\partial\eta},\quad \bar{\Theta}_{,\xi}=\frac{\partial\Theta}{\partial\xi},\quad \bar{\Theta}_{,\eta}=\alpha_1\frac{\partial\Theta}{\partial\eta},\quad \bar{\Theta}_{,\zeta}=\frac{\alpha_1}{\alpha_2}\frac{\partial\Theta}{\partial\zeta},\\ \alpha_1=\frac{a}{b},\quad \alpha_2=\frac{h}{b}.\end{array}\tag{13}$$

Here, following [2, 12], each amplitude functions in (8) are expressed in terms of triplicate series of Chebyshev polynomials multiplied by proper admissible boundary functions, i.e.,

$$\begin{array}{ll}U_1(\xi,\eta,\zeta)=F_{u_1}(\xi,\eta)\sum\limits_{i,j,k=1}^{\infty}A_{ijk}P_i(\xi)P_j(\eta)P_k(\zeta), & \Phi_1(\xi,\eta,\zeta)=F_{\Phi_1}(\xi,\eta)\sum\limits_{\hat{i},\hat{j},\hat{k}=1}^{\infty}\hat{A}_{\hat{i}\hat{j}\hat{k}}P_{\hat{i}}(\xi)P_{\hat{j}}(\eta)P_{\hat{k}}(\zeta),\\ U_2(\xi,\eta,\zeta)=F_{u_2}(\xi,\eta)\sum\limits_{l,m,n=1}^{\infty}B_{lmn}P_l(\xi)P_m(\eta)P_n(\zeta), & \Phi_2(\xi,\eta,\zeta)=F_{\Phi_2}(\xi,\eta)\sum\limits_{\hat{l},\hat{m},\hat{n}=1}^{\infty}\hat{B}_{\hat{l}\hat{m}\hat{n}}P_{\hat{l}}(\xi)P_{\hat{m}}(\eta)P_{\hat{n}}(\zeta),\\ U_3(\xi,\eta,\zeta)=F_{u_3}(\xi,\eta)\sum\limits_{p,q,r=1}^{\infty}C_{pqr}P_p(\xi)P_q(\eta)P_r(\zeta), & \Phi_3(\xi,\eta,\zeta)=F_{\Phi_3}(\xi,\eta)\sum\limits_{\hat{p},\hat{q},\hat{r}=1}^{\infty}\hat{C}_{\hat{p}\hat{q}\hat{r}}P_{\hat{p}}(\xi)P_{\hat{q}}(\eta)P_{\hat{r}}(\zeta),\\ \Theta(\xi,\eta,\zeta)=F_{\Theta}(\xi,\eta)\sum\limits_{\hat{\hat{i}},\hat{\hat{j}},\hat{\hat{k}}=1}^{\infty}\hat{\hat{A}}_{\hat{\hat{i}}\hat{\hat{j}}\hat{\hat{k}}}P_{\hat{\hat{i}}}(\xi)P_{\hat{\hat{j}}}(\eta)P_{\hat{\hat{k}}}(\zeta). & \end{array}\tag{14}$$

One dimensional i^{th} Chebyshev polynomial is given as

$$P_i(\chi)=cos\left[(i-1)\,arccos(\chi)\right].\tag{15}$$

Finally, substituting the series (14) into the energy functional (7) and minimizing this functional with respect to the coefficients of the Chebyshev polynomials, i.e.,

$$\frac{\partial\Pi}{\partial A_{ijk}}=0,\ \frac{\partial\Pi}{\partial B_{lmn}}=0,\ \frac{\partial\Pi}{\partial C_{pqr}}=0,\ \frac{\partial\Pi}{\partial\hat{A}_{\hat{i}\hat{j}\hat{k}}}=0,\ \frac{\partial\Pi}{\partial\hat{B}_{\hat{l}\hat{m}\hat{n}}}=0,\ \frac{\partial\Pi}{\partial\hat{C}_{\hat{p}\hat{q}\hat{r}}}=0,\ \frac{\partial\Pi}{\partial\hat{\hat{A}}_{\hat{\hat{i}}\hat{\hat{j}}\hat{\hat{k}}}}=0,\tag{16}$$

leads to the following eigenvalue problem,

$$\left(\boldsymbol{K}-\Omega^2\,\boldsymbol{M}\right)\boldsymbol{Z}=\boldsymbol{0}\tag{17}$$

with

$$\Omega=\omega\,a\,\sqrt{\rho}.\tag{18}$$

Here, the column vector $\boldsymbol{Z}$ may be written with its sub-column vectors

$$\boldsymbol{Z}=\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{\boldsymbol{C}},\hat{\hat{\boldsymbol{A}}}\}.\tag{19}$$

and each sub-column vector is in the following form [2]

$$\boldsymbol{A}=\{A_{111},\ldots,A_{11K},\ldots,A_{1k1},\ldots,A_{1kK},\ldots,A_{I11},\ldots,A_{IJK}\}.\tag{20}$$

4. Analysis of the spectrum of frequencies and the construction of the optimization problem

Here, we consider a plate made of Gauthier material [16] with the properties $\nu = 0.4$, $E = 5.29\,GPa$, $j = 1.96\times10^{-7}m^2$, $\alpha_1 = 1$, $\alpha_2 = 0.1$. And the plate is considered as simply supported. The number of terms used in equation (14) is taken as $8\times8\times4$ [2]. In addition to the classical constants; the Poisson ratio, ν, and the modulus of elasticity, E, the number of the unknown material coefficients of a linear isotropic microstretch material is seven, namely $\{\kappa,\ \alpha,\ \beta,\ \gamma,\ a_0,\ \lambda_0,\ \lambda_1\}$.

As it is explained above, the existence of the additional frequencies due to the microstructure of the material makes the spectrum different than the one observed in the classical theory. Therefore, an objective function similar to the one used in the classical theory [3–11] could not be sufficient to obtain microstretch elastic properties. Now, to construct a well posed optimization problem, we use two objective functions that should be minimized; the first one is the difference between experimental frequencies and classical frequencies stored in a set (A^{clas}) and the second is the number of the additional frequencies stored in another set (B^{micro}).

The optimization problem may be constructed by superposing two objective functions

$$\underset{subject\ to\ \boldsymbol{X}}{Minimize} \quad \tau_1 \left(\sum_{i=1}^{I} (f_i^e - f_i(\boldsymbol{X}))^2 \right) + \tau_2 \Big(\, length(B^{micro}(\boldsymbol{X})) \Big) . \tag{21}$$

Here, the vector $\boldsymbol{X}$ represents the unknown parameters, i.e.

$$\boldsymbol{X} = \{\kappa,\ \alpha,\ \beta,\ \gamma,\ a_0,\ \lambda_0,\ \lambda_1\}. \tag{22}$$

And, I is the number of the experimental and classical frequencies under consideration, f_i^e is i^{th} experimental frequency, $f_i(\boldsymbol{X})$ is the i^{th} calculated classical frequency corresponding to f_i^e. $B^{micro}(\boldsymbol{X})$ is the set of the additional frequencies due to the microstructure of the medium, τ_1 and τ_2 are the weight functions of the first and second objective functions, respectively and

$$\tau_1 + \tau_2 = 1. \tag{23}$$

The direct search algorithm (DSA) and micro genetic algorithm (mGA) are used to solve inverse optimization problem. The results for Gauthier material [16] is given in the following table [2].

Table 1. Microstretch constants $\{\kappa, \alpha, \beta, \gamma, a_0, \lambda_0, \lambda_1\}$ for Gauthier material [16] obtained from DSA.

$\kappa = 1.3234 \times 10^{-4}\ GPa, \quad \alpha = 8.3255 \times 10^{-2} kN, \quad \beta = 0.10282 kN, \quad \gamma = 3.3349 kN,$ $a_0 = 15.947 kN, \quad \lambda_0 = 0.57702 kPa, \quad \lambda_1 = 34.650 kPa$	
Obtained Freq. (Hz) with above microstretch parameters: 148.534, 353.269, 353.269, 465.074, 465.074, 540.921, 657.715, 658.414, 658.414, 824.744, 825.204	Error 0.0000734863

5. Thermo-microstretch theories

Different theories may be used for thermal problems, such as; static thermoelasticity, uncoupled and coupled quasi-static thermoelasticity, uncoupled and coupled dynamic thermoelasicity etc. Additional approaches may also be given.

As it is well known, the classical theory of heat conduction in solids uses the hypothesis that heat flux is proportional to the gradient of the temperature distribution. As a result of this hypothesis, the corresponding equation appears as a parabolic partial differential equation which results that a thermal disturbance in the body instantaneously affects all points of the body. This infinite speed is contrary with the physical realities. To remove this paradox in the classical theory, generalized thermoelasticity was developed [17]. Generalized theory introduces a short time parameter to the theory to establish a steady state heat conduction when a temperature gradient is suddenly produced in the solid. This is called thermal relaxation time. Thermoelasticity concept has led to a wide range of extension of the classical theory of thermoelasticity. Some theories may be given as [18]:

(i) Coupled thermoelasticity [19].
(ii) Thermoelasticity where the Fourier law of heat conduction is modified by taking into consideration a single relaxation time (Lord and Shulman model) [17].
(iii) Thermoelasticity where the constitutive relations for the stress tensor and the entropy are generalized by introducing two different relaxation times (Green and Lindsay model) [20].
(iv) Thermoelasticity without energy dissipation, proposed by Green and Naghdi [21] where, the Fourier law is replaced by a heat flux rate-temperature gradient relaxation.

These generalizations are the most well-known ones. Their corresponding theories in microstretch theory may be given as follows

1-Couple thermo-microstretch theory (Eringen [1]). The additional terms may be seen in the following equations:

$$\begin{aligned} &t_{kl} = \lambda u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + \kappa(u_{l,k} - \varepsilon_{klm}\phi_r) + \lambda_0\theta\delta_{kl} - \beta_0 T\delta_{kl} \\ &a_0\theta_{,kk} + \tfrac{1}{3}\beta_1 T - \tfrac{1}{3}\lambda_1\theta - \tfrac{1}{3}\lambda_0 u_{k,k} + \rho(l - \tfrac{3}{2}j\ddot{\theta}) = 0 \\ &\rho\gamma_0\dot{T} + \beta_0 T_0\dot{u}_{k,k} + \beta_1 T_0\dot{\theta} - \kappa\nabla^2 T - ph = 0. \end{aligned} \tag{24}$$

And the first field equation may be given in the vector form as:

$$(\lambda + 2\mu + \kappa)\nabla\nabla\cdot\boldsymbol{u} - (\mu+\kappa)\nabla\times\nabla\times\boldsymbol{u} + \kappa\nabla\times\boldsymbol{\phi} + \lambda_0\nabla\theta - \beta_0\nabla T + \rho(\boldsymbol{f} - \ddot{\boldsymbol{u}}) = \boldsymbol{0} \tag{25}$$

2- Corresponding equations to the second theory with one relaxation time are

$$\begin{aligned} &t_{kl} = \lambda u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + \kappa(u_{l,k} - \varepsilon_{klm}\phi_r) + \lambda_0\theta\delta_{kl} - \beta_0 T\delta_{kl}, \\ &a_0\theta_{,kk} + \tfrac{1}{3}\nu_1 T - \tfrac{1}{3}\lambda_1\theta - \tfrac{1}{3}\lambda_0 u_{k,k} + \rho(l - \tfrac{3}{2}j\ddot{\theta}) = 0, \\ &\rho\gamma_0(\dot{T} + \tau_0\ddot{T}) + \nu T_0(\dot{u}_{k,k} + \tau_0\ddot{u}_{k,k}) + \nu_1 T_0(\dot{\theta} - \tau_0\ddot{\theta}) + \kappa\nabla^2 T - ph = 0. \end{aligned} \tag{26}$$

Here

$$\nu = (3\lambda + 2\mu + \kappa)\alpha_{t_1}, \quad \nu_1 = (3\lambda + 2\mu + \kappa)\alpha_{t_2}, \tag{27}$$

and $\alpha_{t_1},\ \alpha_{t_2}$ are the additional coefficients of linear thermal expansion. Then, the corresponding field equation may be written as

$$(\lambda + 2\mu + \kappa)\nabla\nabla\cdot\boldsymbol{u} - (\mu+\kappa)\nabla\times\nabla\times\boldsymbol{u} + \kappa\nabla\times\boldsymbol{\phi} + \lambda_0\nabla\theta - \nu\nabla T + \rho(\boldsymbol{f} - \ddot{\boldsymbol{u}}) = \boldsymbol{0} \tag{28}$$

3- Equations of the coupled theory with two relaxation times are

$$\begin{aligned} &t_{kl} = \lambda u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + \kappa(u_{l,k} - \varepsilon_{klm}\phi_r) + \lambda_0\theta\delta_{kl} - \nu(T + t_1\dot{T})\delta_{kl} \\ &a_0\theta_{,kk} + \tfrac{1}{3}\nu_1(T + t_1\dot{T}) - \tfrac{1}{3}\lambda_1\theta - \tfrac{1}{3}\lambda_0 u_{k,k} + \rho(l - \tfrac{3}{2}j\ddot{\theta}) = 0 \\ &\rho\gamma_0(\dot{T} + \tau_0\ddot{T}) + \nu T_0\dot{u}_{k,k} + \beta_1 T_0(\dot{\theta} - \tau_0\ddot{\theta}) + \kappa\nabla^2 T - ph = 0, \end{aligned} \tag{29}$$

and the same field equation for this case becomes,

$$(\lambda + 2\mu + \kappa)\nabla\nabla\cdot\boldsymbol{u} - (\mu+\kappa)\nabla\times\nabla\times\boldsymbol{u} + \kappa\nabla\times\boldsymbol{\phi} + \lambda_0\nabla\theta - \nu(1 + t_1\frac{\partial}{\partial t})\nabla T + \rho(\boldsymbol{f} - \ddot{\boldsymbol{u}}) = \boldsymbol{0}. \tag{30}$$

As it can be followed from the above expressions, the structure of the equations does not allow us to apply the method used in [2]. With some modifications, second and third theories may be considered, which contain two new unknowns as $\nu(\alpha_{t_1}),\ \nu_1(\alpha_{t_2})$. This part of the work is left to further studies.

6. Propagation of plane waves

To investigate the waves propagating along x_1 direction, we consider the solution of equations (3) with the inclusion of thermal terms as seen in Eq.(24). Here, we take $f_i = l_{ij} = 0$ and $u_i = u_i(x_1, t), \quad \phi_i = \phi_i(x_1, t), \quad \theta = \theta(x_1, t), \quad T = T(x_1, t)$. Then, we find

$$\begin{array}{ll} 1- & (c_1^2 + c_3^2)u_{1,11} + \bar{\lambda}\theta_{,1} - \beta_0^{*2}T = \ddot{u}_1 \\ 2- & (c_2^2 + c_3^2)u_{2,11} - c_3^2\phi_{3,1} = \ddot{u}_2 \\ 3- & (c_2^2 + c_3^2)u_{3,11} - c_3^2\phi_{2,1} = \ddot{u}_3 \\ 4- & (c_4^2 + c_5^2)\phi_{1,11} - 2\omega_0^2\phi_1 = \ddot{\phi}_1 \\ 5- & c_4^2\phi_{2,11} - 2\omega_0^2\phi_2 - \omega_0^2 u_{3,1} = \ddot{\phi}_2 \\ 6- & c_4^2\phi_{3,11} - 2\omega_0^2\phi_3 - \omega_0^2 u_{2,1} = \ddot{\phi}_3 \\ 7- & c_4^2\theta_{,11} - c_7^2\theta - c_8^2 u_{1,1} + \beta_1^{*2}T = \ddot{\theta} \\ 8- & \bar{\gamma}\dot{T} + \beta_0^{*2}\dot{u}_{k,k} + \bar{\beta}_1^2\dot{\theta} - c_3^2\nabla^2 T - h = 0. \end{array} \tag{31}$$

The additional coefficients are

$$\beta_0^{*2} = \frac{\beta_0 T_0}{\rho}, \quad \bar{\beta}_1^2 = \frac{\beta_1 T_0}{\rho}, \quad c_3^2 = \frac{\kappa}{\rho}. \tag{32}$$

Here, we have eight equations for eight unknowns, $u_1, u_2, u_3, \phi_1, \phi_2, \phi_3, \theta$ and T. As it can be followed from above equations, the first, seventh and the last equations are coupled for the unknowns, u_1, θ and T.

Since the direction of propagation is x_1, In the first equation, u_1 represents the longitudinal wave. So, we may call it *longitudinal displacement wave.* The other terms, θ and T may be considered as representing *micro-dilatational* and *thermal* waves respectively.

Second and sixth equations of (31) are also coupled equations for transverse displacement for u_2 and transverse micro-shear ϕ_3. Then, they may be regarded as the equations for representing the propagation of *transverse displacement* and *transverse micro-shear* waves. Third and fifth equations of (31) represent another such pair of coupled equations, specifically showing a coupling between transverse displacement u_3 and transverse micro-shear ϕ_2. And they may be naturally called as the equations for representing the propagation of another couple of *transverse displacement* and *transverse micro-shear* waves in the medium.

To find the dispersion relations for plane harmonic waves, we consider the following forms

$$\begin{array}{ll} u_i = U_i \exp\left[i(\xi x - \omega t)\right], & \phi_i = \Phi_i \exp\left[i(\xi x - \omega\, t)\right] \\ \theta = \Theta \exp\left[i(\xi x - \omega t)\right], & T = \Gamma \exp\left[i(\xi x - \omega\, t)\right] \end{array} \tag{33}$$

where U_i, Φ_i, Θ and Γ are unknown constants, ξ is the wave number, ω is the angular frequency.

a- First, we will consider the uncoupled fourth equation of (31) for ϕ_1. It represents the longitudinal transverse micro shear. To find the dispersion relation for ϕ_1, we use definitions (33) and set the coefficient ϕ_1 to zero, which gives

$$c^2 = \left(\frac{\omega}{\xi}\right)^2 = (c_4^2 + c_5^2)\frac{\omega^2}{\omega^2 - \omega^{*2}}. \tag{34}$$

Here c is the phase velocity and

$$\omega^{*2} = \omega_0^2 = \frac{2\kappa}{\rho j}. \tag{35}$$

Now we may write

$$\begin{array}{ll} \omega > \omega^* & c\ is\ real, \\ \omega = \omega^* & c\ is\ infinite, \\ \omega < \omega^* & c\ is\ imaginary. \end{array} \tag{36}$$

Here ω^* is a cut off frequency which depends on the material characteristic κ and it is positive. Without any micro crack or void in the medium, we arrive the limit case of $\omega^* = 0$ and find $c_1 \Rightarrow \mu^*/\rho_0$ which seems reasonable. As a result, we may conclude that we have here a dispersive wave which propagate only at a frequency higher than ω^*.

b- As an example for coupled equations, we may consider second and fifth equations of (31). These are given for transverse displacement u_2 and transverse micro-shear ϕ_3. They represent the propagation of *transverse displacement* and *transverse micro-shear* waves. Following the same procedure, we find the dispersion equation as,

$$c^4 - c^2 \left[c_2^2 + 2c_3^2 + c_4^2 + 2c^{*2}\right] + (c_2^2 + c_3^2)(c_2^2 + 2c^{*2}) = 0. \tag{37}$$

The investigation of this equation is similar to the one given in [13]. Thus, we shall not give the details here.

c- Last group of coupled equations are given for u_1, θ and T representing *longitudinal displacement* wave, *longitudinal micro-dilatational* and *thermal* waves respectively. Substituting above definitions for u_1, θ and T into the corresponding equations, we obtain

$$\begin{array}{l} \left[\omega^2 - (c_1^2 + c_3^2)\, \xi^2\right] U + i\xi\bar{\lambda}\Theta + \bar{\beta}_0\,(i\xi)\,\Gamma = 0 \\ -c_8^2 i\xi U + \left[3\omega^2 - (\xi^2 c_6^2 + c_7^2)\right]\Theta + \omega_1^2\Gamma = 0 \\ \beta_0^{*2}\omega\xi U - \beta_1^{*2} i\omega\Theta - \left(c_3^2\xi^2 - \gamma\omega i\right)\Gamma = 0. \end{array} \tag{38}$$

To obtain nonzero solutions for the unknowns $\boldsymbol{U}$, Θ, Γ in the above system of equations, the determinant of the coefficient matrix must be zero. This gives the dispersion relation for the propagation of longitudinal displacement wave, longitudinal micro-dilatational and thermal waves.

Since it is rather complicated, we would like to consider a more simple problem just to see the effects of the thermal field. Considering

$$\beta_1 << \lambda_0 < \beta_0, \tag{39}$$

we may neglect the small terms in the first step in the corresponding equations and take

$$T = \left(\frac{\varepsilon}{\xi}\right) U. \tag{40}$$

Here

$$\varepsilon = \frac{\beta_0 T_0}{\kappa}. \tag{41}$$

Now, substituting this approximate form of T in first and eight equations of (31), we obtain the following dispersion relation,

$$\omega^4 - [\alpha + \varepsilon]\,\omega^2 - \beta + \gamma\varepsilon = 0. \tag{42}$$

Here

$$\alpha = \left(\frac{1}{3}\right) c_6^2\xi^2 + \left(\frac{1}{3}\right) c_7^2 + \left(c_1^2 + c_3^2\right), \quad \beta = \bar{\lambda}_0 c_8^2\xi^2, \quad \gamma = \left(\frac{1}{3}\right)\left(c_6^2\xi^2 + c_7^2\right). \tag{43}$$

It is important to see here whether ω^2 is real. For this condition the radical must be nonzero, which gives,

$$\begin{array}{l} \frac{1}{9}\left(c_6^2\xi^2 + c_7^2\right)^2 + \left(c_1^2 + c_3^2\right)^2 + \frac{2}{3}\left(c_6^2\xi^2 + c_7^2\right)\left(c_1^2 + c_3^2\right) + \\ 2\left[\frac{1}{3}\left(c_6^2\xi^2 + c_7^2\right) + \left(c_1^2 + c_3^2\right)\right]\varepsilon + \varepsilon^2 + 4\beta - \frac{4}{3}\left(c_6^2\xi^2 + c_7^2\right)\varepsilon > 0. \end{array} \tag{44}$$

To satisfy this condition for all values of ξ, we must have

$$\begin{array}{l} 1.\quad c_6^4 > 0, \\ 2.\quad \frac{1}{3}c_7^2\left(c_1^2 + c_3^2\right) + 6\lambda_0 \frac{c_8^2}{c_6^2} > \varepsilon, \\ 3.\ \left[\frac{1}{3}c_7^2 + \left(c_1^2 + c_3^2\right)\right]^2 + 2\left(c_1^2 + c_3^2\right)\varepsilon + \varepsilon^2 > 0. \end{array} \tag{45}$$

Here, the first and third conditions are obviously satisfied. A condition similar to the second one is also obtained by Eringen for the microstretch bodies by comparing the dispersion relation with the corresponding one in lattice dynamics [1].

7. Conclusions

The upper bounds of microstretch elastic constants of a microstretch material are found by constructing an optimization problem with two objective functions. So, these results may be used in damaged bodies modelled by microstretch theory. An example with the results obtsined here is given in a recent work of the authors and found the displacement field of a damaged hollow cylinder [22].

In the second part of this work, a discussion is given for the investigation of thermal problems of microstructured materials with different approaches. Later, the propagation of plane waves for thermo-microstretch material is studied. Determination of the microthermal constants is left to further studies.

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