Nonlinear Dynamics Volume 82, Issue 1, October 2015, pp 73-84

# On head-on collision between two solitary waves in shallow water: The use of the extended PLK method

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**Abstract**

In the present work, we examined the head-on-collision of solitary waves in shallow water theory, through the use of extended Poincare-Lighthill-Kuo(PLK) method based on the combination of reductive perturbation method with strained coordinates. Motivated with the result obtained by Ozden and Demiray (Int.J. Nonlinear Mech., 69:66-70, 2015), we introduced a set of stretched coordinates that include some unknown functions which are to be determined so as to remove secularities that might occur in the solution. By expanding these unknown functions and the field variables into power series in the smallness parameter $\epsilon$, introducing them into the field equations and imposing the conditions to remove the secularities we obtained some evolution equations. By seeking a progressive wave solution to these evolution equations we determined the speed correction terms and the phase shift functions. The result obtained here is exactly the same with found by Ozden and Demiray (Int.J. Nonlinear Mech.69:66-70, 2015), wherein the analysis employed by Su and Mirie (J. Fluid Mech., 98:09-525, 1980) is utilized.

**Keywords:** Head-on collision, extended PLK method, phase shifts

# 1 Introduction

It is well-known that long-time asymptotic behaviour of two dimensional unidirectional shallow water waves in the case of weak nonlinearity is described by the Korteweg-de Vries (KdV) equation [1]. Since, the inverse scattering transform (IST) for exactly solving the KdV equation was found by Gardner, Kruskal and Miura [2], the interesting features of the collision between solitary waves had been revealed: When two solitary waves approach closely, they interact, exchange their energies and position with one another, and, then separate off, regaining their original forms. Throughout the whole process of the collision, the solitary waves are remarkably stable entities preserving their identities through the interaction. The unique effect due to the collision is their phase shifts. It is believed that this striking colliding property of solitary waves can only be preserved in a conservative system.

According to IST, all the KdV solitary waves travel in the same direction, under the boundary conditions vanishing at infinity [2, 3]; so for overtaking collision between solitary waves, one can use the IST to obtain the overtaking colliding effect of solitary waves. However, for the head-on collision between solitary waves, one must employ some kind of asymptotic expansion to solve the original field equations. In this regard, for the study of head-on-collision problems, a comprehensive approach had been presented by Su and Mirie [4], in which the Poincare-Lighthill-Kuo(PLK) method had been employed. To determine the unknown phase shift functions, in their analysis they made the statement that "*although certain terms do not cause any secularity at this order but they will cause secularity at the higher order expansion, therefore, those terms must vanish*". Several researchers, utilizing the implication of this statement studied the head-on-collision of solitary wave problems in various media [5-19]. Unfortunately, our calculations for the higher order expansion show that the terms mentioned in their work do not cause any secularity in the solution. Our results had been justified by one of the authors (Su, private communication). Therefore, the result they found for the phase shift functions is incorrect. The details of our arguments are given in [20], in which the same kind of solution method have been utilized.

In the present work, we study the same problem through the use of extended PLK method, in which the classical reductive perturbation method is combined with the strained coordinates. Motivated with the result obtained in [20] we introduce the strained coordinates as

$$\begin{aligned} \epsilon^{1/2}(x-t) &= \xi + \epsilon p(\tau) + \epsilon^2 P(\xi,\eta,\tau), \\ \epsilon^{1/2}(x+t) &= \eta + \epsilon q(\tau) + \epsilon^2 Q(\xi,\eta,\tau), \qquad \tau = \epsilon^{3/2} t, \end{aligned}$$

where $\epsilon$ is the smallness parameter measuring the weakness of dispersion and nonlinearity, $p(\tau)$ and $q(\tau)$ are two unknown functions characterizing the higher order dispersive effects, $P(\xi,\eta,\tau)$ and $Q(\xi,\eta,\tau)$ are two unknown functions characterizing the phase shifts after collision. These unknown functions are to be determined from the higher order perturbation expansions so as to remove possible secularities that occur in the solution.

Expanding the field variables and these unknown functions into power

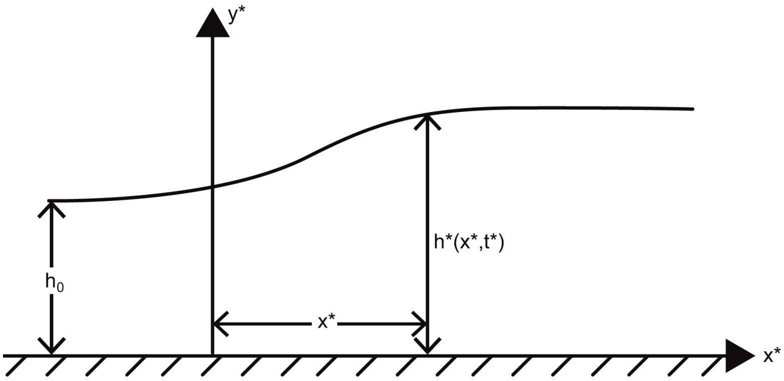


Figure 1:

series of $\epsilon$, introducing these expansion into the field equations and setting the coefficients of various powers of $\epsilon$ equal to zero we obtained a set of partial differential equations. By solving these differential equations and removing possible secularities that occur in the solution we obtained various evolution equations and restrictions that make it possible to determine the unknown functions. Seeking a progressive wave solution to these evolution equations we obtained the velocity correction terms and the phase shifts. It is observed that the result found here is exactly the same with one obtained in [20].

## 2 Basic Equations

We consider a plane irrotational flow of an incompressible fluid. Let $\phi^\star(x^\star, y^\star, t^\star)$ be the velocity potential related to the velocity components $u^\star$ and $v^\star$ in the $x^\star$ and $y^\star$ directions, respectively, by

$$u^\star = \frac{\partial \phi^\star}{\partial x^\star}, \qquad v^\star = \frac{\partial \phi^\star}{\partial y^\star}. \tag{1}$$

The incompressibility of the fluid requires that $\phi^\star$ must satisfy the Laplace equation

$$\frac{\partial^2 \phi^\star}{\partial x^{\star 2}} + \frac{\partial^2 \phi^\star}{\partial y^{\star 2}} = 0. \tag{2}$$

The boundary conditions to be satisfied are:

$$\frac{\partial \phi^\star}{\partial y^\star} = 0 \quad \text{at } y^\star = 0,$$
$$\frac{\partial \phi^\star}{\partial y^\star} = \frac{\partial h^\star}{\partial t^\star} + \frac{\partial \phi^\star}{\partial x^\star}\frac{\partial h^\star}{\partial x^\star} \quad \text{at } y^\star = h^\star,$$

$$\frac{\partial \phi^\star}{\partial t^\star} + \frac{1}{2}\left[\left(\frac{\partial \phi^\star}{\partial x^\star}\right)^2 + \left(\frac{\partial \phi^\star}{\partial y^\star}\right)^2\right] + g(h^\star - h_0) = 0$$

$$\text{at } y^\star = h^\star, \tag{3}$$

where $g$ is gravity acceleration of the earth and $h_0$ is the water still level(equilibrium depth).

At this stage it is convenient to introduce the following nondimensional quantities

$$x^\star = h_0 x, \quad y^\star = h_0 y, \quad t^\star = \left(\frac{h_0}{g}\right)^{1/2} t,$$

$$h^\star = h_0(1 + \hat{\zeta}), \quad \phi^\star = \left(g h_0^3\right)^{1/2} \hat{\phi} \tag{4}$$

Introducing (4) into the equations (2)-(3), the following nondimensional equations are obtained

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0, \tag{5}$$

$$\frac{\partial \hat{\phi}}{\partial y} = 0 \quad \text{at } y = 0,$$

$$\frac{\partial \hat{\phi}}{\partial y} = \frac{\partial \hat{\zeta}}{\partial t} + \frac{\partial \hat{\phi}}{\partial x}\frac{\partial \hat{\zeta}}{\partial x} \quad \text{at } y = 1 + \hat{\zeta},$$

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \hat{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \hat{\phi}}{\partial y}\right)^2\right] + \hat{\zeta} = 0$$

$$\text{at } y = 1 + \hat{\zeta}. \tag{6}$$

These equations will be used as we study the head-on collision problem in shallow water theory.

Assuming a polynomial solution for $\hat{\phi}$ in terms of $y$, the solution satisfying the boundary condition $\frac{\partial \hat{\phi}}{\partial y} = 0$ at $y = 0$ may be given by

$$\hat{\phi} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\partial^{2n} \hat{\Phi}}{\partial x^{2n}} y^{2n}. \tag{7}$$

where $\hat{\Phi}(x,t)$ is the value of $\hat{\phi}(x,y,t)$ at $y = 0$. This solution must satisfy the last two boundary conditions in (6).

$$\frac{\partial\hat{\zeta}}{\partial t}+\frac{\partial}{\partial x}\Bigg\{(1+\hat{\zeta})\hat{w}$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\hat{\zeta})^{2n+1}}{(2n+1)!}\frac{\partial^{2n}\hat{w}}{\partial x^{2n}}\Bigg\}=0, \tag{8}$$

$$\frac{\partial\hat{w}}{\partial t}+\frac{\partial}{\partial x}\Bigg\{\hat{\zeta}+\frac{\hat{w}^2}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\hat{\zeta})^{2n}}{(2n)!}$$
$$\left[\frac{\partial^{2n}\hat{w}}{\partial t\partial x^{2n-1}}+\frac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\frac{\partial^m\hat{w}}{\partial x^m}\frac{\partial^{2n-m}\hat{w}}{\partial x^{2n-m}}\right]\Bigg\}$$
$$=0, \tag{9}$$

where $w=\dfrac{\partial\hat{\phi}}{\partial x}$ and $\dbinom{2n}{m}$ is the binomial coefficient.

# 3 Extended PLK Method

For our future purposes, we introduce the following stretched coordinates

$$\epsilon^{\frac{1}{2}}(x-t)=\xi+\epsilon p(\tau)+\epsilon^2P(\xi,\eta,\tau),$$
$$\epsilon^{\frac{1}{2}}(x+t)=\eta+\epsilon q(\tau)+\epsilon^2Q(\xi,\eta,\tau),$$
$$\epsilon^{3/2}t=\tau, \tag{10}$$

where $\epsilon$ is the smallness parameter measuring the weakness of dispersion and nonlinearity, $p(\tau)$ and $q(\tau)$ are two unknown functions characterizing the higher order dispersive effects, $P(\xi,\eta,\tau)$ and $Q(\xi,\eta,\tau)$ are two unknown functions characterizing the phase shifts after collision. Then, the following differential relations hold true

$$\frac{\partial}{\partial x}=\frac{\epsilon^{\frac{1}{2}}}{D}\Bigg\{\left[1+\epsilon^2\left(\frac{\partial Q}{\partial\eta}-\frac{\partial P}{\partial\eta}\right)\right]\frac{\partial}{\partial\xi}$$
$$+\left[1+\epsilon^2\left(\frac{\partial P}{\partial\xi}-\frac{\partial Q}{\partial\xi}\right)\right]\frac{\partial}{\partial\eta}\Bigg\},$$
$$\frac{\partial}{\partial t}=\epsilon^{\frac{1}{2}}\Bigg\{\epsilon\frac{\partial}{\partial\tau}-\frac{1}{D}\Bigg[1+\epsilon^2\left(\frac{dp}{d\tau}+\frac{\partial P}{\partial\eta}+\frac{\partial Q}{\partial\eta}\right)$$
$$+\epsilon^3\frac{\partial P}{\partial\tau}+\epsilon^4\left(\frac{dp}{d\tau}\frac{\partial Q}{\partial\eta}-\frac{dq}{d\tau}\frac{\partial P}{\partial\eta}\right)$$

$$+\epsilon^5\left(\frac{\partial P}{\partial \tau}\frac{\partial Q}{\partial \eta}-\frac{\partial Q}{\partial \tau}\frac{\partial P}{\partial \eta}\right)\bigg]\frac{\partial}{\partial \xi}$$
$$+\frac{1}{D}\bigg[1+\epsilon^2\left(-\frac{dq}{d\tau}+\frac{\partial P}{\partial \xi}+\frac{\partial Q}{\partial \xi}\right)$$
$$-\epsilon^3\frac{\partial Q}{\partial \tau}+\epsilon^4\left(\frac{dp}{d\tau}\frac{\partial Q}{\partial \xi}-\frac{dq}{d\tau}\frac{\partial P}{\partial \xi}\right)$$
$$+\epsilon^5\left(\frac{\partial P}{\partial \tau}\frac{\partial Q}{\partial \xi}-\frac{\partial Q}{\partial \tau}\frac{\partial P}{\partial \xi}\right)\bigg]\frac{\partial}{\partial \eta}\Bigg\} \tag{11}$$

where $D$ is defined by

$$D=\left(1+\epsilon^2\frac{\partial P}{\partial \xi}\right)\left(1+\epsilon^2\frac{\partial Q}{\partial \eta}\right)-\epsilon^4\frac{\partial P}{\partial \eta}\frac{\partial Q}{\partial \xi}. \tag{12}$$

We assume that the field quantities $\hat{w}$, $\hat{\zeta}$, $p(\tau)$, $q(\tau)$, $P(\xi,\eta,\tau)$ and $Q(\xi,\eta,\tau)$ can be expanded into asymptotic series in $\epsilon$ as

$$\begin{aligned}
&\hat{w}=\epsilon\left[w_0+\epsilon w_1+\epsilon^2 w_2+\epsilon^3 w_3+\epsilon^4 w_4+...\right],\\
&\hat{\zeta}=\epsilon\left[\zeta_0+\epsilon\zeta_1+\epsilon^2\zeta_2+\epsilon^3\zeta_3+\epsilon^4\zeta_4+...\right],\\
&p(\tau)=p_0(\tau)+\epsilon p_1(\tau)+\epsilon^2 p_2(\tau)+\epsilon^3 p_3(\tau)+...\ ,\\
&q(\tau)=q_0(\tau)+\epsilon q_1(\tau)+\epsilon^2 q_2(\tau)+\epsilon^3 q_3(\tau)+...\ ,\\
&P(\xi,\eta,\tau)=P_0(\xi,\eta,\tau)+\epsilon P_1(\xi,\eta,\tau)+...\ ,\\
&Q(\xi,\eta,\tau)=Q_0(\xi,\eta,\tau)+\epsilon Q_1(\xi,\eta,\tau)+...\ .
\end{aligned} \tag{13}$$

Inserting (11) and (13) into equations (8) and (9) and setting the coefficients of like powers of $\epsilon$ equal to zero the following equations are obtained

## $\mathcal{O}(\epsilon)$ *equations:*

$$\begin{aligned}
&\frac{\partial \zeta_0}{\partial \eta}-\frac{\partial \zeta_0}{\partial \xi}+\frac{\partial w_0}{\partial \eta}+\frac{\partial w_0}{\partial \xi}=0,\\
&\frac{\partial \zeta_0}{\partial \eta}+\frac{\partial \zeta_0}{\partial \xi}+\frac{\partial w_0}{\partial \eta}-\frac{\partial w_0}{\partial \xi}=0,
\end{aligned} \tag{14}$$

## $\mathcal{O}(\epsilon^2)$ *equations:*

$$\frac{\partial \zeta_1}{\partial \eta}-\frac{\partial \zeta_1}{\partial \xi}+\frac{\partial w_1}{\partial \eta}+\frac{\partial w_1}{\partial \xi}+\frac{\partial \zeta_0}{\partial \tau}+\frac{\partial}{\partial \eta}(\zeta_0 w_0)$$

$$
\begin{aligned}
&+\frac{\partial}{\partial \xi}(\zeta_0 w_0)-\frac{1}{6}\left(\frac{\partial^3 w_0}{\partial \xi^3}+3\frac{\partial^3 w_0}{\partial \xi^2\partial \eta}+3\frac{\partial^3 w_0}{\partial \xi\partial \eta^2}\right.\\
&\left.+\frac{\partial^3 w_0}{\partial \eta^3}\right)=0,\\
&\frac{\partial \zeta_1}{\partial \eta}+\frac{\partial \zeta_1}{\partial \xi}+\frac{\partial w_1}{\partial \eta}-\frac{\partial w_1}{\partial \xi}+\frac{\partial w_0}{\partial \tau}+\frac{1}{2}\frac{\partial}{\partial \eta}(w_0^2)\\
&+\frac{1}{2}\frac{\partial}{\partial \xi}(w_0^2)+\frac{1}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3}+\frac{\partial^3 w_0}{\partial \xi^2\partial \eta}-\frac{\partial^3 w_0}{\partial \xi\partial \eta^2}\right.\\
&\left.-\frac{\partial^3 w_0}{\partial \eta^3}\right)=0, \qquad (15)
\end{aligned}
$$

$\mathcal{O}\left(\epsilon^3\right)$ *equations:*

$$
\begin{aligned}
&\frac{\partial \zeta_2}{\partial \eta}-\frac{\partial \zeta_2}{\partial \xi}+\frac{\partial w_2}{\partial \eta}+\frac{\partial w_2}{\partial \xi}+\frac{\partial \zeta_1}{\partial \tau}+\frac{\partial}{\partial \eta}(\zeta_1 w_0)\\
&+\frac{\partial}{\partial \xi}(\zeta_1 w_0)+\frac{\partial}{\partial \eta}(\zeta_0 w_1)+\frac{\partial}{\partial \xi}(\zeta_0 w_1)\\
&-\frac{1}{6}\left(\frac{\partial^3 w_1}{\partial \xi^3}+3\frac{\partial^3 w_1}{\partial \xi^2\partial \eta}+3\frac{\partial^3 w_1}{\partial \xi\partial \eta^2}+\frac{\partial^3 w_1}{\partial \eta^3}\right)\\
&-\frac{dq_0}{d\tau}\frac{\partial \zeta_0}{\partial \eta}-\frac{dp_0}{d\tau}\frac{\partial \zeta_0}{\partial \xi}+\frac{1}{120}\left(\frac{\partial^5 w_0}{\partial \xi^5}+5\frac{\partial^5 w_0}{\partial \xi^4\partial \eta}\right.\\
&\left.+10\frac{\partial^5 w_0}{\partial \xi^3\partial \eta^2}+10\frac{\partial^5 w_0}{\partial \xi^2\partial \eta^3}+5\frac{\partial^5 w_0}{\partial \xi\partial \eta^4}+\frac{\partial^5 w_0}{\partial \eta^5}\right)\\
&-\frac{\zeta_0}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3}+3\frac{\partial^3 w_0}{\partial \xi^2\partial \eta}+3\frac{\partial^3 w_0}{\partial \xi\partial \eta^2}+\frac{\partial^3 w_0}{\partial \eta^3}\right)\\
&-\left(6\frac{\partial P_0}{\partial \xi}+7\frac{\partial Q_0}{\partial \eta}-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \xi}(\zeta_0-w_0)\\
&+\frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(\zeta_0-w_0)-\frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(\zeta_0+w_0)\\
&+\left(7\frac{\partial P_0}{\partial \xi}+6\frac{\partial Q_0}{\partial \eta}-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(\zeta_0+w_0)\\
&-\frac{1}{2}\left(\frac{\partial^2 w_0}{\partial \xi^2}+2\frac{\partial^2 w_0}{\partial \xi\partial \eta}+\frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial \zeta_0}{\partial \eta}\right.\\
&\left.+\frac{\partial \zeta_0}{\partial \xi}\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial \zeta_2}{\partial \eta}+\frac{\partial \zeta_2}{\partial \xi}+\frac{\partial w_2}{\partial \eta}-\frac{\partial w_2}{\partial \xi}+\frac{\partial w_1}{\partial \tau}+\frac{\partial}{\partial \eta}(w_0 w_1) \\
&+\frac{\partial}{\partial \xi}(w_0 w_1)-\frac{dq_0}{d\tau}\frac{\partial w_0}{\partial \eta}-\frac{dp_0}{d\tau}\frac{\partial w_0}{\partial \xi}+\frac{1}{2}\left(\frac{\partial^3 w_1}{\partial \xi^3}\right. \\
&\left.+\frac{\partial^3 w_1}{\partial \xi^2 \partial \eta}-\frac{\partial^3 w_1}{\partial \xi \partial \eta^2}-\frac{\partial^3 w_1}{\partial \eta^3}\right)-\frac{1}{24}\left(\frac{\partial^5 w_0}{\partial \xi^5}\right. \\
&+3\frac{\partial^5 w_0}{\partial \xi^4 \partial \eta}+2\frac{\partial^5 w_0}{\partial \xi^3 \partial \eta^2}-2\frac{\partial^5 w_0}{\partial \xi^2 \partial \eta^3}-3\frac{\partial^5 w_0}{\partial \xi \partial \eta^4} \\
&\left.-\frac{\partial^5 w_0}{\partial \eta^5}\right)+\frac{1}{2}\left(\frac{\partial^2 w_0}{\partial \xi^2}+2\frac{\partial^2 w_0}{\partial \xi \partial \eta}+\frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial w_0}{\partial \eta}\right. \\
&\left.+\frac{\partial w_0}{\partial \xi}\right)-\frac{w_0}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3}+3\frac{\partial^3 w_0}{\partial \xi^2 \partial \eta}+3\frac{\partial^3 w_0}{\partial \xi \partial \eta^2}\right. \\
&\left.+\frac{\partial^3 w_0}{\partial \eta^3}\right)+\zeta_0\left(\frac{\partial^3 w_0}{\partial \xi^3}+\frac{\partial^3 w_0}{\partial \xi^2 \partial \eta}-\frac{\partial^3 w_0}{\partial \xi \partial \eta^2}\right. \\
&\left.-\frac{\partial^3 w_0}{\partial \eta^3}\right)-\frac{1}{2}\frac{\partial}{\partial \tau}\left[\frac{\partial^2 w_0}{\partial \xi^2}+2\frac{\partial^2 w_0}{\partial \xi \partial \eta}+\frac{\partial^2 w_0}{\partial \eta^2}\right] \\
&-\frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(\zeta_0-w_0)-\frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(\zeta_0+w_0)+\left(7\frac{\partial P_0}{\partial \xi}\right. \\
&\left.+6\frac{\partial Q_0}{\partial \eta}-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(\zeta_0+w_0)+\left(6\frac{\partial P_0}{\partial \xi}\right. \\
&\left.+7\frac{\partial Q_0}{\partial \eta}-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \xi}(\zeta_0-w_0)+\left(\frac{\partial^2 w_0}{\partial \xi^2}\right. \\
&\left.-\frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial \zeta_0}{\partial \eta}+\frac{\partial \zeta_0}{\partial \xi}\right)=0. \qquad (16)
\end{aligned}
$$

## 3.1 Solution of the field equations

From the solution of the set (14) we have

$$
\begin{aligned}
\zeta_0 &= f(\xi,\tau)+g(\eta,\tau), \\
w_0 &= f(\xi,\tau)-g(\eta,\tau), \qquad (17)
\end{aligned}
$$

where $f(\xi,\tau)$ and $g(\eta,\tau)$ are two unknown functions whose governing equations will be obtained later.

The solution of (15) yields

$$
2\frac{\partial}{\partial \eta}(\zeta_1+w_1)+2\frac{\partial f}{\partial \tau}+3f\frac{\partial f}{\partial \xi}+\frac{1}{3}\frac{\partial^3 f}{\partial \xi^3}-g\frac{\partial g}{\partial \eta}
$$

$$+\frac{2}{3}\frac{\partial^3 g}{\partial\eta^3}-f\frac{\partial g}{\partial\eta}-\frac{\partial f}{\partial\xi}g=0, \tag{18}$$

$$2\frac{\partial}{\partial\xi}(\zeta_1-w_1)-2\frac{\partial g}{\partial\tau}+3g\frac{\partial g}{\partial\eta}+\frac{1}{3}\frac{\partial^3 g}{\partial\eta^3}-f\frac{\partial f}{\partial\xi}$$
$$+\frac{2}{3}\frac{\partial^3 f}{\partial\xi^3}-g\frac{\partial f}{\partial\xi}-\frac{\partial g}{\partial\eta}f=0. \tag{19}$$

Integrating (18) with respect to $\eta$ and (19) with respect to $\xi$ we obtain

$$2(\zeta_1+w_1)+\eta\left[2\frac{\partial f}{\partial\tau}+3f\frac{\partial f}{\partial\xi}+\frac{1}{3}\frac{\partial^3 f}{\partial\xi^3}\right]-\frac{g^2}{2}$$
$$+\frac{2}{3}\frac{\partial^2 g}{\partial\eta^2}-fg-M(\eta,\tau)\frac{\partial f}{\partial\xi}=4F_1(\xi,\tau), \tag{20}$$
$$2(\zeta_1-w_1)-\xi\left[2\frac{\partial g}{\partial\tau}-3g\frac{\partial g}{\partial\eta}-\frac{1}{3}\frac{\partial^3 g}{\partial\eta^3}\right]-\frac{f^2}{2}$$
$$+\frac{2}{3}\frac{\partial^2 f}{\partial\xi^2}-fg-N(\xi,\tau)\frac{\partial g}{\partial\eta}=4G_1(\eta,\tau), \tag{21}$$

where $F_1(\xi,\tau)$ and $G_1(\eta,\tau)$ are new unknown functions, $M(\eta,\tau)$ and $N(\xi,\tau)$ are defined by

$$M(\eta,\tau)=\int^{\eta} g(\eta^{'},\tau)d\eta^{'},\ N(\xi,\tau)=\int^{\xi} f(\xi^{'},\tau)d\xi^{'}. \tag{22}$$

At first glance, it is seen that the terms proportional to $\xi$ and $\eta$ cause secularity. In order to remove the secularities we must have

$$\frac{\partial f}{\partial\tau}+\frac{3}{2}f\frac{\partial f}{\partial\xi}+\frac{1}{6}\frac{\partial^3 f}{\partial\xi^3}=0, \tag{23}$$
$$\frac{\partial g}{\partial\tau}-\frac{3}{2}g\frac{\partial g}{\partial\eta}-\frac{1}{6}\frac{\partial^3 g}{\partial\eta^3}=0. \tag{24}$$

These are Korteweg-de Vries equations. The solution of equations (20) and (21) for $\zeta_1$ and $w_1$ gives

$$\zeta_1=F_1(\xi,\tau)+G_1(\eta,\tau)+\frac{1}{4}M(\eta,\tau)\frac{\partial f}{\partial\xi}$$
$$+\frac{1}{4}N(\xi,\tau)\frac{\partial g}{\partial\eta}+\frac{1}{8}\left(f^2+g^2\right)+\frac{1}{2}fg$$
$$-\frac{1}{6}\left(\frac{\partial^2 f}{\partial\xi^2}+\frac{\partial^2 g}{\partial\eta^2}\right), \tag{25}$$

$$
\begin{aligned}
w_1 = {} & F_1(\xi,\tau) - G_1(\eta,\tau) + \frac{1}{4}M(\eta,\tau)\frac{\partial f}{\partial \xi} \\
& - \frac{1}{4}N(\xi,\tau)\frac{\partial g}{\partial \eta} - \frac{1}{8}\left(f^2 - g^2\right) \\
& + \frac{1}{6}\left(\frac{\partial^2 f}{\partial \xi^2} - \frac{\partial^2 g}{\partial \eta^2}\right).
\end{aligned} \tag{26}
$$

Su and Mirie [4] stated that, although the terms $M(\eta,\tau)\dfrac{\partial f}{\partial \xi}$ and $N(\xi,\tau)\dfrac{\partial g}{\partial \eta}$ in (25) and (26) do not cause any secularity but they will cause secularity in the next order perturbation expansion. However, in what follows it will be shown that it is not the case.

The localized progressive wave solution for the KdV equations (23) and (24) yield

$$
f = A\,\mathrm{sech}^2\,\zeta_+ \ , \qquad \zeta_+ = \left(\frac{3A}{4}\right)^{1/2}\left(\xi - \frac{A}{2}\tau\right), \tag{27}
$$

$$
g = B\,\mathrm{sech}^2\,\zeta_- \ , \qquad \zeta_- = \left(\frac{3B}{4}\right)^{1/2}\left(\eta + \frac{B}{2}\tau\right), \tag{28}
$$

where $A$ and $B$ are the amplitudes of the solitary waves. Substituting (17), (25) and (26) into the set of equations (16), we obtain

$$
\begin{aligned}
& 2\frac{\partial}{\partial \eta}(\zeta_2 + w_2) + 2\frac{\partial F_1}{\partial \tau} + 3\frac{\partial}{\partial \xi}(fF_1) + \frac{1}{3}\frac{\partial^3 F_1}{\partial \xi^3} \\
& - \frac{3}{8}f^2\frac{\partial f}{\partial \xi} + \frac{1}{12}f\frac{\partial^3 f}{\partial \xi^3} + \frac{11}{12}\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi^2} - 2\frac{dp_0}{d\tau}\frac{\partial f}{\partial \xi} \\
& + \frac{1}{45}\frac{\partial^5 f}{\partial \xi^5} - \frac{1}{2}\frac{\partial^3 f}{\partial \tau \partial \xi^2} - f\frac{\partial G_1}{\partial \eta} - \frac{\partial f}{\partial \xi}G_1 - \frac{\partial}{\partial \eta}(gG_1) \\
& + \frac{2}{3}\frac{\partial^3 G_1}{\partial \eta^3} - F_1\frac{\partial g}{\partial \eta} - g\frac{\partial F_1}{\partial \xi} - \frac{1}{4}\frac{\partial^2 f}{\partial \xi^2}gM - \frac{1}{4}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta}M \\
& + \left(\frac{1}{6}\frac{\partial^4 g}{\partial \eta^4} - \frac{1}{4}\frac{\partial}{\partial \eta}\left(g\frac{\partial g}{\partial \eta}\right) - \frac{1}{4}f\frac{\partial^2 g}{\partial \eta^2} - \frac{1}{4}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta}\right)N \\
& + \frac{1}{4}fg\frac{\partial g}{\partial \eta} + \left(\frac{1}{8}f^2 - \frac{1}{6}\frac{\partial^2 f}{\partial \xi^2}\right)\frac{\partial g}{\partial \eta} + \frac{11}{6}f\frac{\partial^3 g}{\partial \eta^3} \\
& + \left(\frac{3}{4}f\frac{\partial f}{\partial \xi} + \frac{13}{12}\frac{\partial^3 f}{\partial \xi^3}\right)g + \frac{5}{12}\frac{\partial f}{\partial \xi}\frac{\partial^2 g}{\partial \eta^2} + \frac{3}{8}g^2\frac{\partial g}{\partial \eta} \\
& - 4\frac{\partial P_0}{\partial \eta}\frac{\partial f}{\partial \xi} + \frac{29}{12}\frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2} + \frac{4}{3}g\frac{\partial^3 g}{\partial \eta^3} + \frac{4}{45}\frac{\partial^5 g}{\partial \eta^5} = 0,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&2\frac{\partial}{\partial\xi}(\zeta_2-w_2)-2\frac{\partial G_1}{\partial\tau}+3\frac{\partial}{\partial\eta}(gG_1)+\frac{1}{3}\frac{\partial^3 G_1}{\partial\eta^3}\\
&-\frac{3}{8}g^2\frac{\partial g}{\partial\eta}+\frac{1}{12}g\frac{\partial^3 g}{\partial\eta^3}+\frac{11}{12}\frac{\partial g}{\partial\eta}\frac{\partial^2 g}{\partial\eta^2}+2\frac{dq_0}{d\tau}\frac{\partial g}{\partial\eta}\\
&+\frac{1}{45}\frac{\partial^5 g}{\partial\eta^5}+\frac{1}{2}\frac{\partial^3 g}{\partial\tau\partial\eta^2}-g\frac{\partial F_1}{\partial\xi}-F_1\frac{\partial g}{\partial\eta}-\frac{\partial}{\partial\xi}(fF_1)\\
&+\frac{2}{3}\frac{\partial^3 F_1}{\partial\xi^3}-\frac{\partial f}{\partial\xi}G_1-f\frac{\partial G_1}{\partial\eta}-\frac{1}{4}\frac{\partial g}{\partial\eta}fN-\frac{1}{4}\frac{\partial g}{\partial\eta}\frac{\partial f}{\partial\xi}N\\
&+\left(\frac{1}{6}\frac{\partial^4 f}{\partial\xi^4}-\frac{1}{4}\frac{\partial}{\partial\xi}\left(f\frac{\partial f}{\partial\xi}\right)-\frac{1}{4}g\frac{\partial^2 f}{\partial\xi^2}-\frac{1}{4}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}\right)M\\
&+\frac{1}{4}gf\frac{\partial f}{\partial\xi}+\left(\frac{1}{8}g^2-\frac{1}{6}\frac{\partial^2 g}{\partial\eta^2}\right)\frac{\partial f}{\partial\xi}+\frac{11}{6}g\frac{\partial^3 f}{\partial\xi^3}\\
&+\left(\frac{3}{4}g\frac{\partial g}{\partial\eta}+\frac{13}{12}\frac{\partial^3 g}{\partial\eta^3}\right)f+\frac{5}{12}\frac{\partial g}{\partial\eta}\frac{\partial^2 f}{\partial\xi^2}+\frac{3}{8}f^2\frac{\partial f}{\partial\xi}\\
&-4\frac{\partial Q_0}{\partial\xi}\frac{\partial g}{\partial\eta}+\frac{29}{12}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}+\frac{4}{3}f\frac{\partial^3 f}{\partial\xi^3}\\
&+\frac{4}{45}\frac{\partial^5 f}{\partial\xi^5}=0. \qquad (30)
\end{aligned}
$$

Integrating (29) with respect to $\eta$ and (30) with respect to $\xi$ we obtain

$$
\begin{aligned}
&2(\zeta_2+w_2)+\eta\left(2\frac{\partial F_1}{\partial\tau}+3\frac{\partial}{\partial\xi}(fF_1)+\frac{1}{3}\frac{\partial^3 F_1}{\partial\xi^3}\right.\\
&-\frac{3}{8}f^2\frac{\partial f}{\partial\xi}+\frac{1}{12}f\frac{\partial^3 f}{\partial\xi^3}+\frac{11}{12}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}-2\frac{dp_0}{d\tau}\frac{\partial f}{\partial\xi}\\
&\left.+\frac{1}{45}\frac{\partial^5 f}{\partial\xi^5}-\frac{1}{2}\frac{\partial^3 f}{\partial\tau\partial\xi^2}\right)-(f+g)G_1-gF_1\\
&-\frac{\partial f}{\partial\xi}\int^{\eta}G_1 d\eta'+\frac{2}{3}\frac{\partial^2 G_1}{\partial\eta^2}+\left(\frac{3}{4}f\frac{\partial f}{\partial\xi}+\frac{13}{12}\frac{\partial^3 f}{\partial\xi^3}\right.\\
&\left.-\frac{\partial F_1}{\partial\xi}\right)M(\eta,\tau)+\left(\frac{1}{6}\frac{\partial^3 g}{\partial\eta^3}-\frac{1}{4}f\frac{\partial g}{\partial\eta}-\frac{1}{4}g\frac{\partial f}{\partial\xi}\right.\\
&\left.-\frac{1}{4}g\frac{\partial g}{\partial\eta}\right)N(\xi,\tau)-\frac{1}{4}\frac{\partial^2 f}{\partial\xi^2}\int^{\eta}gM d\eta'\\
&-\frac{1}{4}\frac{\partial f}{\partial\xi}\int^{\eta}\left(\frac{\partial g}{\partial\eta}M\right)d\eta'+\frac{1}{8}fg^2+\frac{1}{8}f^2 g-\frac{1}{6}\frac{\partial^2 f}{\partial\xi^2}g\\
&+\frac{5}{12}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}+\frac{1}{8}g^3+\frac{11}{6}f\frac{\partial^2 g}{\partial\eta^2}+\frac{4}{3}g\frac{\partial^2 g}{\partial\eta^2}
\end{aligned}
$$

$$+\frac{13}{24}\left(\frac{\partial g}{\partial \eta}\right)^2+\frac{4}{45}\frac{\partial^4 g}{\partial \eta^4}-4P_0\frac{\partial f}{\partial \xi}=4F_2(\xi,\tau), \tag{31}$$

$$\begin{aligned}
&2(\zeta_2-w_2)+\xi\left(-2\frac{\partial G_1}{\partial \tau}+3\frac{\partial}{\partial \eta}(gG_1)+\frac{1}{3}\frac{\partial^3 G_1}{\partial \eta^3}\right.\\
&-\frac{3}{8}g^2\frac{\partial g}{\partial \eta}+\frac{1}{12}g\frac{\partial^3 g}{\partial \eta^3}+\frac{11}{12}\frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2}+2\frac{dq_0}{d\tau}\frac{\partial g}{\partial \eta}\\
&\left.+\frac{1}{45}\frac{\partial^5 g}{\partial \eta^5}+\frac{1}{2}\frac{\partial^3 g}{\partial \tau \partial \eta^2}\right)-(f+g)F_1-fG_1\\
&-\frac{\partial g}{\partial \eta}\int^{\xi}F_1 d\xi'+\frac{2}{3}\frac{\partial^2 F_1}{\partial \xi^2}+\left(\frac{3}{4}g\frac{\partial g}{\partial \eta}+\frac{13}{12}\frac{\partial^3 g}{\partial \eta^3}\right.\\
&\left.-\frac{\partial G_1}{\partial \eta}\right)N(\xi,\tau)+\left(\frac{1}{6}\frac{\partial^3 f}{\partial \xi^3}-\frac{1}{4}g\frac{\partial f}{\partial \xi}-\frac{1}{4}f\frac{\partial g}{\partial \eta}\right.\\
&\left.-\frac{1}{4}f\frac{\partial f}{\partial \xi}\right)M(\eta,\tau)-\frac{1}{4}\frac{\partial^2 g}{\partial \eta^2}\int^{\xi}fNd\xi'\\
&-\frac{1}{4}\frac{\partial g}{\partial \eta}\int^{\xi}\left(\frac{\partial f}{\partial \xi}N\right)d\xi'+\frac{1}{8}fg^2+\frac{1}{8}f^2g+\frac{11}{6}\frac{\partial^2 f}{\partial \xi^2}g\\
&+\frac{5}{12}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta}+\frac{1}{8}f^3-\frac{1}{6}f\frac{\partial^2 g}{\partial \eta^2}+\frac{4}{3}f\frac{\partial^2 f}{\partial \xi^2}\\
&+\frac{13}{24}\left(\frac{\partial f}{\partial \xi}\right)^2+\frac{4}{45}\frac{\partial^4 f}{\partial \xi^4}-4Q_0\frac{\partial g}{\partial \eta}=4G_2(\eta,\tau)
\end{aligned} \tag{32}$$

where $F_2(\xi,\tau)$ and $G_2(\eta,\tau)$ are two unknown functions whose evolution equations will be obtained from next order equations. Again the terms proportional to $\xi$ and $\eta$ in these equations cause to secularity in the solution. In order to remove secularity, the coefficient of $\eta$ in (31) and the coefficient of $\xi$ in (32) must vanish, that is

$$\begin{aligned}
\frac{\partial F_1}{\partial \tau}+\frac{3}{2}\frac{\partial}{\partial \xi}(fF_1)+\frac{1}{6}\frac{\partial^3 F_1}{\partial \xi^3} &= \frac{3}{16}f^2\frac{\partial f}{\partial \xi}+\frac{1}{4}\frac{\partial^3 f}{\partial \tau \partial \xi^2}\\
&-\frac{11}{24}\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi^2}+\frac{dp_0}{d\tau}\frac{\partial f}{\partial \xi}\\
&-\frac{1}{90}\frac{\partial^5 f}{\partial \xi^5}-\frac{1}{24}f\frac{\partial^3 f}{\partial \xi^3},
\end{aligned} \tag{33}$$

$$\frac{\partial G_1}{\partial \tau} - \frac{3}{2}\frac{\partial}{\partial \eta}(gG_1) - \frac{1}{6}\frac{\partial^3 G_1}{\partial \eta^3} = -\frac{3}{16}g^2\frac{\partial g}{\partial \eta} + \frac{1}{4}\frac{\partial^3 g}{\partial \tau \partial \eta^2} + \frac{11}{24}\frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2} + \frac{dq_0}{d\tau}\frac{\partial g}{\partial \eta} + \frac{1}{90}\frac{\partial^5 g}{\partial \eta^5} + \frac{1}{24}g\frac{\partial^3 g}{\partial \eta^3} \tag{34}$$

As is seen from the equations (31) and (32) the other terms in the expression of $\zeta_2$ and $w_2$ do not cause any secularity for this order, but it might be possible to have secularities in the next order. Seeking a progressive wave solution for the equations (33) and (34) of the form $F_1 = F_1(\zeta_+)$, $G_1 = G_1(\zeta_-)$ the following equations are obtained

$$\frac{A}{8}F_1^{''} + \frac{1}{2}(3f - A)F_1 = \left(\frac{dp_0}{d\tau} - \frac{19A^2}{40}\right) f + \frac{9}{16}Af^2 + \frac{1}{8}f^3, \tag{35}$$

$$-\frac{B}{8}G_1^{''} + \frac{1}{2}(B - 3g)\,G_1 = \left(\frac{dq_0}{d\tau} + \frac{19B^2}{40}\right) g - \frac{9}{16}Bg^2 - \frac{1}{8}g^3. \tag{36}$$

The first terms in the right-hand side cause to secularity; therefore the coefficients of $f$ and $g$ must vanish, which yields

$$p_0 = \frac{19}{40}A^2\tau, \qquad q_0 = -\frac{19}{40}B^2\tau. \tag{37}$$

Then the particular solution of the differential equations (35) and (36) may be given by

$$F_1 = Af - \frac{1}{8}f^2, \qquad G_1 = Bg - \frac{1}{8}g^2. \tag{38}$$

By using the above results one can obtain the following identities for the terms involving the functions $g$, $G_1$ and $M$

$$\int g^2 d\eta^{'} = \frac{M}{3}(g + 2B), \quad \int G_1 d\eta^{'} = \frac{M}{24}(22B - g),$$

$$\int gM d\eta^{'} = -\frac{2}{3}g, \quad \int \left(\frac{\partial g}{\partial \eta}M\right) d\eta^{'} = \frac{2M}{3}(g - B). \tag{39}$$

Similar expressions are valid for the terms involving $f$, $F_1$ and $N$. Then the equations (31) and (32) may be written in the following form

$$\begin{aligned}\zeta_2 + w_2 = {} & \frac{1}{16}g^3 + \frac{43}{16}Bg^2 - \frac{7}{5}B^2 g + \left(\frac{A}{2} - \frac{9B}{4}\right) fg \\ & + 4fg^2 - \frac{1}{8}f^2 g - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[-\frac{B}{4}\frac{\partial g}{\partial \eta}\right. \\ & \left.+\frac{7}{8}g\frac{\partial g}{\partial \eta} + \frac{1}{8}f\frac{\partial g}{\partial \eta} + \frac{1}{8}\frac{\partial f}{\partial \xi}g\right] N \\ & + \left[\left(-\frac{9A}{8} + \frac{3B}{8}\right)\frac{\partial f}{\partial \xi} + \frac{35}{8}f\frac{\partial f}{\partial \xi}\right. \\ & \left.+\frac{1}{16}\frac{\partial f}{\partial \xi}g\right] M + 2P_0\frac{\partial f}{\partial \xi} + 2F_2(\xi, \tau), \end{aligned} \tag{40}$$

$$\begin{aligned}\zeta_2 - w_2 = {} & \frac{1}{16}f^3 + \frac{43}{16}Af^2 - \frac{7}{5}A^2 f + \left(\frac{B}{2} - \frac{9A}{4}\right) fg \\ & + 4f^2 g - \frac{1}{8}fg^2 - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[-\frac{A}{4}\frac{\partial f}{\partial \xi}\right. \\ & \left.+\frac{7}{8}f\frac{\partial f}{\partial \xi} + \frac{1}{8}g\frac{\partial f}{\partial \xi} + \frac{1}{8}f\frac{\partial g}{\partial \eta}\right] M \\ & + \left[\left(-\frac{9B}{8} + \frac{3A}{8}\right)\frac{\partial g}{\partial \eta} + \frac{35}{8}g\frac{\partial g}{\partial \eta}\right. \\ & \left.+\frac{1}{16}f\frac{\partial g}{\partial \eta}\right] N + 2Q_0\frac{\partial g}{\partial \eta} + 2G_2(\eta, \tau). \end{aligned} \tag{41}$$

In obtaining the equations (40) and (41) we have utilized the following identities

$$\begin{aligned} \frac{\partial^2 f}{\partial \xi^2} &= 3Af - \frac{9}{2}f^2, \quad \left(\frac{\partial f}{\partial \xi}\right)^2 = 3Af^2 - 3f^3, \\ \frac{\partial^4 f}{\partial \xi^4} &= \frac{135}{2}f^3 - \frac{135}{2}Af^2 + 9A^2 f, \\ \frac{\partial^6 f}{\partial \xi^6} &= \frac{-8505}{4}f^4 + 2835Af^3 - \frac{1701}{2}A^2 f^2 \\ & \quad + 27A^3 f. \end{aligned} \tag{42}$$

Similar identities can also be obtained for the derivatives of the function $g$.

As might be seen from equations (40) and (41) these terms appearing in the expressions of $\zeta_1$ and $w_1$ do not cause any secularity in the solution of $\zeta_2$

and $w_2$. Therefore the statement by Su and Mirie [4] is incorrect. However as we stated before, some of the terms appearing in the expressions of $\zeta_2$ and $w_2$ (The equations (40) and (41) ) may cause additional secularity in the expressions of $\zeta_3$ and $w_3$. There appears to be two types of secularity in the solution of $\mathcal{O}(\epsilon^4)$ equation. As was seen before, the first type of secularity results from the terms proportional to $\xi$ and $\eta$ which will be studied later. The second type secularity occurs from the terms proportional $\int^{\xi} N(\xi', \tau) d\xi'$ and $\int^{\eta} M(\eta', \tau) d\eta'$ as $\xi(\eta) \to \pm\infty$. Here we shall first only consider the parts of $\mathcal{O}(\epsilon^4)$ equations leading to $\int^{\eta} M(\eta', \tau) d\eta'$ type of secularity. Similar expressions may be valid for $\int^{\xi} N(\xi', \tau) d\xi'$ type of secularity.

For this purpose we consider the following part of the $\mathcal{O}(\epsilon^4)$ equation

$$\begin{aligned}
&2\frac{\partial}{\partial \eta}(\zeta_3 + w_3) + \frac{\partial}{\partial \tau}(\zeta_2 + w_2) + \frac{1}{6}\frac{\partial^3}{\partial \xi^3}(\zeta_2 + w_2) \\
&+ \frac{3}{4}\frac{\partial}{\partial \xi}\left[(\zeta_0 + w_0)(\zeta_2 + w_2)\right] - \frac{dp_0}{d\tau}\frac{\partial}{\partial \xi}(\zeta_1 + w_1) \\
&+ \frac{1}{2}\frac{\partial^2 w_0}{\partial \xi^2}\frac{\partial}{\partial \xi}(\zeta_1 + w_1) + \frac{1}{2}\frac{\partial}{\partial \xi}(\zeta_0 + w_0)\frac{\partial^2 w_1}{\partial \xi^2} \\
&- \frac{1}{30}\frac{\partial^5 w_1}{\partial \xi^5} - \frac{1}{2}\frac{\partial^3 w_1}{\partial \tau \partial \xi^2} + w_1\frac{\partial}{\partial \xi}(\zeta_1 + w_1) + \zeta_1\frac{\partial w_1}{\partial \xi} \\
&- \frac{1}{2}\frac{\partial}{\partial \xi}\left[\zeta_0(\zeta_2 - w_2)\right] - \frac{1}{6}\frac{\partial^3}{\partial \xi^3}(\zeta_2 - w_2) = 0.
\end{aligned} \tag{43}$$

A similar expression may be given for $2\frac{\partial}{\partial \xi}(\zeta_3 - w_3)$ equation. We split (43) into two parts which contain the variables $\zeta_2 + w_2$ and $(\zeta_1,\ w_1,\ \zeta_2 - w_2)$, respectively. Then, we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(\zeta_2 + w_2) + \frac{3}{4}\frac{\partial}{\partial \xi}\left[(\zeta_0 + w_0)(\zeta_2 + w_2)\right] \\
&+ \frac{1}{6}\frac{\partial^3}{\partial \xi^3}(\zeta_2 + w_2) = \\
&\frac{35}{16}\left[\frac{189}{4}f^4 - 63Af^3 + 18A^2f^2\right]M,
\end{aligned} \tag{44}$$

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial^2 w_0}{\partial\xi^2}\frac{\partial}{\partial\xi}(\zeta_1+w_1)-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(\zeta_1+w_1)\\
&+\frac{1}{2}\frac{\partial}{\partial\xi}(\zeta_0+w_0)\frac{\partial^2 w_1}{\partial\xi^2}-\frac{1}{30}\frac{\partial^5 w_1}{\partial\xi^5}-\frac{1}{2}\frac{\partial^3 w_1}{\partial\tau\partial\xi^2}\\
&+w_1\frac{\partial}{\partial\xi}(\zeta_1+w_1)+\zeta_1\frac{\partial w_1}{\partial\xi}-\frac{1}{2}\frac{\partial}{\partial\xi}\left[\zeta_0\left(\zeta_2-w_2\right)\right]\\
&-\frac{1}{6}\frac{\partial^3}{\partial\xi^3}(\zeta_2-w_2)=\\
&\frac{1}{16}\left[\frac{189}{4}f^4-63Af^3+18A^2f^2\right]M,
\end{aligned}
\tag{45}
$$

where we have used the identities given by (39) and (42). As is seen from the equations (44) and (45), the terms proportional to $M(\eta,\tau)$ do not vanish and they cause the secularity of the type $\int^{\eta} M(\eta^{'},\tau)d\eta^{'}$ in the expression of $\zeta_3$ and $w_3$. Similar expression may be given for $\int^{\xi} N(\xi^{'},\tau)d\xi^{'}$ type of secularities.

By direct substitution in the expressions of $\zeta_2+w_2$ and $\zeta_2-w_2$

$$
P_0=-\frac{9}{4}f(\xi,\tau)M(\eta,\tau),\quad Q_0=-\frac{9}{4}g(\eta,\tau)N(\xi,\tau) \tag{46}
$$

these secularities may be removed. These expressions make it possible to determine phase shift functions.

To obtain the secularities of type $\eta$ (or $\xi$) we use the following part of the $\mathcal{O}(\epsilon^4)$ equation to obtain the governing equation for $F_2(\xi,\tau)$

$$
\begin{aligned}
&2\frac{\partial}{\partial\eta}(\zeta_3+w_3)+\frac{\partial}{\partial\tau}(\zeta_2+w_2)+w_0\frac{\partial}{\partial\xi}(\zeta_2+w_2)\\
&+\frac{\partial w_0}{\partial\xi}(\zeta_2+w_2)+\frac{\partial}{\partial\xi}(\zeta_0 w_2)+\frac{1}{3}\frac{\partial^3 w_2}{\partial\xi^3}\\
&-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(\zeta_1+w_1)+\frac{1}{2}\frac{\partial}{\partial\xi}\left[\frac{\partial^2 w_0}{\partial\xi^2}\zeta_1\right]+\frac{\partial}{\partial\xi}(\zeta_1 w_1)\\
&+\frac{1}{2}\frac{\partial}{\partial\xi}\left[\frac{\partial w_0}{\partial\xi}\frac{\partial w_1}{\partial\xi}\right]-\frac{1}{30}\frac{\partial^5 w_1}{\partial\xi^5}+\frac{1}{2}\frac{\partial}{\partial\xi}\left[\zeta_0\frac{\partial^2 w_1}{\partial\xi^2}\right]\\
&-\frac{1}{2}\frac{\partial^3 w_1}{\partial\xi^2\partial\tau}-\frac{1}{2}w_0\frac{\partial^3 w_1}{\partial\xi^3}-\frac{1}{2}\frac{\partial^3 w_0}{\partial\xi^3}w_1+w_1\frac{\partial w_1}{\partial\xi}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\partial w_0}{\partial \xi}\right)^2\frac{\partial \zeta_0}{\partial \xi}-\frac{1}{8}\frac{\partial^4 w_0}{\partial \xi^4}\frac{\partial \zeta_0}{\partial \xi}-\frac{dp_1}{d\tau}\frac{\partial}{\partial \xi}(\zeta_0+w_0)\\
&-\frac{\partial^2 w_0}{\partial \xi \partial \tau}\frac{\partial \zeta_0}{\partial \xi}-w_0\frac{\partial^2 w_0}{\partial \xi^2}\frac{\partial \zeta_0}{\partial \xi}+\frac{\partial w_0}{\partial \xi}\frac{\partial^2 w_0}{\partial \xi^2}\zeta_0\\
&-\frac{1}{8}\frac{\partial^5 w_0}{\partial \xi^5}\zeta_0-\frac{\partial^3 w_0}{\partial \xi^2 \partial \tau}\zeta_0-w_0\frac{\partial^3 w_0}{\partial \xi^3}\zeta_0+\frac{1}{2}\frac{dp_0}{d\tau}\frac{\partial^3 w_0}{\partial \xi^3}\\
&+\frac{1}{12}\frac{\partial^2 w_0}{\partial \xi^2}\frac{\partial^3 w_0}{\partial \xi^3}-\frac{1}{8}\frac{\partial w_0}{\partial \xi}\frac{\partial^4 w_0}{\partial \xi^4}+\frac{1}{840}\frac{\partial^7 w_0}{\partial \xi^7}\\
&+\frac{1}{24}\frac{\partial^5 w_0}{\partial \xi^4 \partial \tau}+\frac{1}{24}w_0\frac{\partial^5 w_0}{\partial \xi^5}=0. \qquad (47)
\end{aligned}
$$

We substitute the field variables into (47) then the terms proprtional to $\eta$ in this equation cause to secularity. In order to remove secularity, the coefficient of $\eta$ in (47) must vanish, that is

$$
\frac{\partial F_2}{\partial \tau}+\frac{3}{2}\frac{\partial}{\partial \xi}(fF_2)+\frac{1}{6}\frac{\partial^3 F_2}{\partial \xi^3}=\frac{\partial S(f)}{\partial \xi} \qquad (48)
$$

where $S(f)$ is defined as follows

$$
\begin{aligned}
S(f)=&\left(\frac{dp_1}{d\tau}-\frac{55}{112}A^3\right)f-\left(\frac{393}{320}+\frac{3B}{16A}\right)A^2f^2\\
&+\left(\frac{201}{32}+\frac{3B}{16A}\right)Af^3-\frac{591}{128}f^4. \qquad (49)
\end{aligned}
$$

Seeking a progressive wave solution for the equation (48) of the form $F_2 = F_2(\zeta_+)$, the following solution is obtained

$$
\begin{aligned}
F_2 =& \frac{197}{160}f^3-\left(\frac{217}{160}+\frac{3B}{16A}\right)Af^2\\
&+\left(\frac{43}{40}+\frac{B}{8A}\right)A^2f,\\
p_1(\tau) =& \frac{55}{112}A^3\tau. \qquad (50)
\end{aligned}
$$

Similarly, for other unknowns we have

$$
\begin{aligned}
G_2 =& \frac{197}{160}g^3-\left(\frac{217}{160}+\frac{3A}{16B}\right)Bg^2\\
&+\left(\frac{43}{40}+\frac{A}{8B}\right)B^2g,
\end{aligned}
$$

$$q_1(\tau) = -\frac{55}{112} B^3 \tau. \tag{51}$$

Then, the final solution for $\zeta_2$ and $w_2$ take the following form

$$\begin{aligned}
\zeta_2 = & \frac{101}{80}(f^3 + g^3) + \frac{31}{16}(fg^2 + f^2 g) \\
& - \frac{1}{80}(Af^2 + Bg^2) - \frac{3}{16}(Bf^2 + Ag^2) \\
& - \frac{7}{8}(A + B)fg + \frac{3}{8}(A^2 f + B^2 g) + \frac{1}{8}AB(f + g) \\
& - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[\left(\frac{-11A + 3B}{16}\right)\frac{\partial f}{\partial \xi} + \frac{3}{8}f\frac{\partial f}{\partial \xi}\right. \\
& \left. + \frac{1}{16}f\frac{\partial g}{\partial \eta} + \frac{3}{32}\frac{\partial f}{\partial \xi}g\right] M + \left[\left(\frac{3A - 11B}{16}\right)\frac{\partial g}{\partial \eta}\right. \\
& \left. + \frac{3}{8}g\frac{\partial g}{\partial \eta} + \frac{1}{16}\frac{\partial f}{\partial \xi}g + \frac{3}{32}f\frac{\partial g}{\partial \eta}\right] N,
\end{aligned} \tag{52}$$

$$\begin{aligned}
w_2 = & \frac{6}{5}(f^3 - g^3) + \frac{33}{16}(fg^2 - f^2 g) \\
& - \frac{27}{10}(Af^2 - Bg^2) - \frac{3}{16}(Bf^2 - Ag^2) \\
& + \frac{11}{8}(A - B)fg + \frac{71}{40}(A^2 f - B^2 g) + \frac{AB}{8}(f - g) \\
& + \left[\left(\frac{-7A + 3B}{16}\right)\frac{\partial f}{\partial \xi} - \frac{1}{2}f\frac{\partial f}{\partial \xi} - \frac{1}{16}f\frac{\partial g}{\partial \eta}\right. \\
& \left. - \frac{1}{32}\frac{\partial f}{\partial \xi}g\right] M + \left[\left(\frac{-3A + 7B}{16}\right)\frac{\partial g}{\partial \eta}\right. \\
& \left. + \frac{1}{2}g\frac{\partial g}{\partial \eta} + \frac{1}{16}\frac{\partial f}{\partial \xi}g + \frac{1}{32}f\frac{\partial g}{\partial \eta}\right] N.
\end{aligned} \tag{53}$$

Thus, for this order the trajectories of the solitary waves become

$$\begin{aligned}
\epsilon^{\frac{1}{2}}(x - t) &= \xi + \epsilon p_0 + \epsilon^2 (p_1 + P_0) + \mathcal{O}(\epsilon^3), \\
\epsilon^{\frac{1}{2}}(x + t) &= \eta + \epsilon q_0 + \epsilon^2 (q_1 + Q_0) + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{54}$$

To obtain the phase shifts after a head-on collision of solitary waves characterized by $A$ and $B$ are asymptotically far from each other at the initial time ($t = -\infty$), the solitary wave $A$ is at $\xi = 0$, $\eta = -\infty$, and the solitary wave $B$ is at $\eta = 0$, $\xi = +\infty$, respectively. After the collision ($t = +\infty$), the solitary wave $B$ is far to the right of solitary wave $A$, i.e., the solitary wave

$A$ is at $\xi = 0$, $\eta = +\infty$, and the solitary wave $B$ is at $\eta = 0$, $\xi = -\infty$. Using (46) and (54) one can obtain the corresponding phase shifts $\Delta_A$ and $\Delta_B$ as follows:

$$\begin{aligned}
\Delta_A &= \epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=\infty} - \epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=-\infty} \\
&= -\epsilon^2 \frac{9}{4} f(0) \int_{-\infty}^{+\infty} g(\eta')d\eta' \\
&= -\epsilon^2 \frac{9A}{4} \int_{-\infty}^{+\infty} g(\eta')d\eta' \qquad (55)
\end{aligned}$$

$$\begin{aligned}
\Delta_B &= \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=-\infty} - \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=\infty} \\
&= \epsilon^2 \frac{9}{4} g(0) \int_{-\infty}^{+\infty} f(\xi')d\xi' \\
&= \epsilon^2 \frac{9B}{4} \int_{-\infty}^{+\infty} f(\xi')d\xi'. \qquad (56)
\end{aligned}$$

Using the explicit expressions of $f(\xi)$ and $g(\eta)$ the phase shifts are obtained as

$$\Delta_A = -\epsilon^2 3\sqrt{3} A B^{1/2}, \quad \Delta_B = \epsilon^2 3\sqrt{3} A^{1/2} B. \qquad (57)$$

Here, as opposed to the results of previous works on the same subject the phase shifts depend on the amplitudes of both waves.

# 4 Summary of the result of section 3

In section 3, we have obtained the following results

$$\begin{aligned}
\hat{\zeta}(f,g) =& \epsilon \Bigg\{ (f+g) + \epsilon \Bigg( \frac{3}{4}(f^2+g^2) + \frac{1}{2}(Af+Bg) \\
& + \frac{1}{2}fg + \frac{1}{4}M(\eta,\tau)\frac{\partial f}{\partial \xi} + \frac{1}{4}N(\xi,\tau)\frac{\partial g}{\partial \eta} \Bigg) \\
& + \epsilon^2 \Bigg( \frac{101}{80}(f^3+g^3) - \frac{1}{80}(Af^2+Bg^2) \\
& - \frac{3}{16}(Bf^2+Ag^2) + \frac{3}{8}(A^2f+B^2g) \\
& + \frac{1}{8}AB(f+g) + \frac{31}{16}(fg^2+f^2g)
\end{aligned}$$

$$-\frac{7}{8}(A+B)fg - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta}$$
$$+\left[\left(\frac{-11A+3B}{16}\right)\frac{\partial f}{\partial \xi} + \frac{3}{8}f\frac{\partial f}{\partial \xi}\right.$$
$$\left.+\frac{1}{16}f\frac{\partial g}{\partial \eta} + \frac{3}{32}\frac{\partial f}{\partial \xi}g\right] M$$
$$+\left[\left(\frac{3A-11B}{16}\right)\frac{\partial g}{\partial \eta} + \frac{3}{8}g\frac{\partial g}{\partial \eta} + \frac{1}{16}\frac{\partial f}{\partial \xi}g\right.$$
$$\left.\left.\left.+\frac{3}{32}f\frac{\partial g}{\partial \eta}\right] N\right) + \ldots\right\}. \quad (58)$$

Similar expression may be given for $\hat{w}(f, g)$.

$$p(\tau) = \epsilon\left(\frac{19}{40}A^2 + \epsilon\frac{55}{112}A^3\right)\tau, \quad (59)$$

$$q(\tau) = \epsilon\left(-\frac{19}{40}B^2 - \epsilon\frac{55}{112}B^3\right)\tau, \quad (60)$$

$$P = -\frac{9}{4}f(\xi,\tau)\int_{-\infty}^{\eta} g(\eta',\tau)d\eta', \quad (61)$$

$$Q = -\frac{9}{4}g(\eta,\tau)\int_{\infty}^{\xi} f(\xi',\tau)d\xi', \quad (62)$$

and

$$\zeta_+ = \left(\frac{3A\epsilon}{4}\right)^{\frac{1}{2}}\left[x - c_R t + \frac{9}{4}\epsilon^{\frac{3}{2}}f(\xi,\tau)\right.$$
$$\left.\times\int_{-\infty}^{\eta} g(\eta',\tau)d\eta'\right], \quad (63)$$

$$\zeta_- = \left(\frac{3B\epsilon}{4}\right)^{\frac{1}{2}}\left[x + c_L t + \frac{9}{4}\epsilon^{\frac{3}{2}}g(\eta,\tau)\right.$$
$$\left.\times\int_{\infty}^{\xi} f(\xi',\tau)d\xi'\right], \quad (64)$$

where $c_R$ and $c_L$ are defined by

$$c_R = 1 + \left(\epsilon\frac{A}{2} + \epsilon^2\frac{19}{40}A^2 + \epsilon^3\frac{55}{112}A^3\right), \quad (65)$$

$$c_L = 1 + \left(\epsilon\frac{B}{2} + \epsilon^2\frac{19}{40}B^2 + \epsilon^3\frac{55}{112}B^3\right). \tag{66}$$

The equations (61) and (62) serve to define the phase changes. Before the collision

$$\eta \to -\infty, \quad P \to 0, \qquad \xi \to \infty, \quad Q \to 0 \tag{67}$$

and after the collision

$$\eta \to \infty, \qquad P = -9A\left(\frac{B}{3}\right)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_+, \tag{68}$$

$$\xi \to -\infty, \qquad Q = 9B\left(\frac{A}{3}\right)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_-. \tag{69}$$

In this section we shall illustrate the profiles of right-going waves before and after the collision. For that purpose we set $g(\eta,\tau) = 0$ in the expression $\hat{\zeta}$ and obtain

$$\begin{aligned}\hat{\zeta} =& \epsilon\left\{f + \epsilon\left(\frac{3}{4}f^2 + \frac{A}{2}f\right) + \epsilon^2\left(\frac{101}{80}f^3 - \frac{A}{80}f^2\right.\right.\\ &\left.\left.-\frac{3B}{16}f^2 + \frac{3A^2}{8}f + \frac{AB}{8}f\right) + \ldots\right\}\end{aligned} \tag{70}$$

with

$$f = A\operatorname{sech}^2\left[\left(\frac{3A\epsilon}{4}\right)^{\frac{1}{2}}(x - c_R t + \Theta)\right] \tag{71}$$

where

$$\Theta = \epsilon^{\frac{3}{2}} 9A\left(\frac{B}{3}\right)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_+. \tag{72}$$

The variations of the wave profiles for surface elevation parameter $\hat{\zeta}$ before the collision ($\Theta = 0$) and after the collision ($\Theta$ is given as in (72)) are depicted in Figure 2, for various values of parameters $\epsilon$, $A$ and $B$. As is seen from the figure the wave profile before the collision is symmetric, whereas after the collision it is unsymmetrical and tilts backward with respect to the direction of its propagation.

# 5 Conclusion

Utilizing the non-dimensionalized equations derived by Su and Mirie [4] and introducing a set of stretched coordinates that include some unknown functions which are to be determined from the removal of possible secularities in

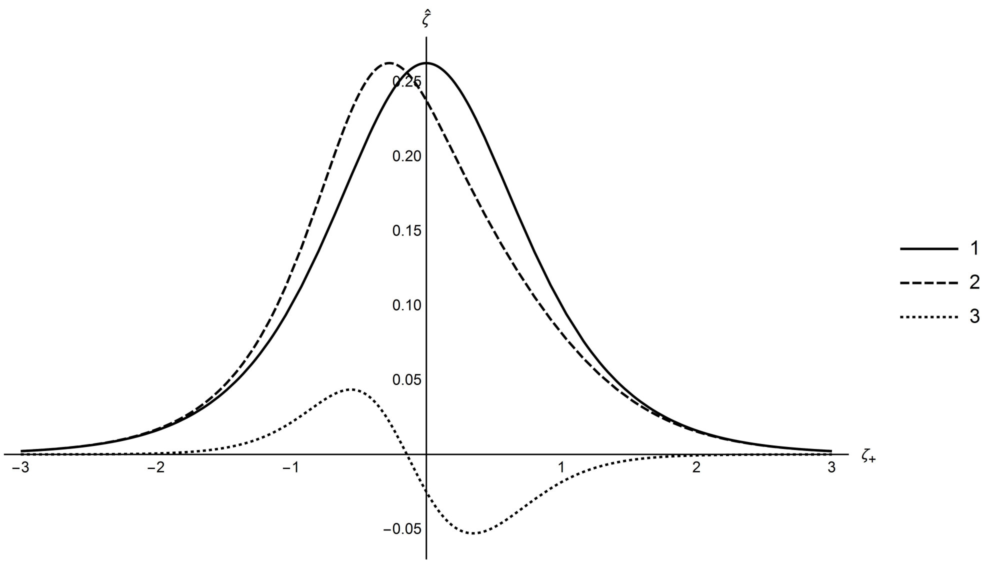


Figure 2: Right-going wave profile $\hat{\zeta}$ for $\epsilon = 0.4$, $A = B = 0.5$. 1: before collision; 2: after collision; 3: difference between the wave profiles before and after the collision.

the solution, we have studied the head-on-collision of solitary waves in shallow water theory. Expanding these unknown functions and the field variables into power series of the smallness parameter $\epsilon$ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations. By solving these differential equations and imposing the requirements for the removal of possible secularities we obtained the speed of correction terms and the phase shift functions. Our calculations show that the present results are exactly the same with those found in [20], whereas it is totally different from the results of Su and Mirie [4]. The variations of the wave profiles for right-going wave ($\hat{\zeta}$) before and after the collision are illustrated in Fig. 2. As is seen from the figure the wave profile is symmetric before the collision whereas it is unsymmetrical after the collision with tilts backward with respect to the direction of its propagation.

# 6 Compliance with Ethical Standards

We certify that any part of this work was not published or submitted for publication elsewhere, and we do not have any conflict of interest with anybody else.

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