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Quantum Fisher Information of a 3x3 Bound Entangled State and Its Relation with Geometric Discord

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Abstract Recent studies on quantum Fisher information (QFI) have been focused mostly on qubit systems within the context of how entanglement helps surpassing the classical limit of separable states and the limit that a given entangled system can achieve for parameter estimation. However, there are only a few works on bound entangled systems. In this work, we study the QFI of a system of the smallest dimension that bound entanglement can be observed: A bipartite quantum system of two particles of three-levels each. An interesting property of this state is that depending only on a parameter, the state can be separable, bound entangled or free entangled. We show that QFI exhibits a smooth and continues increase with respect to this parameter throughout the transition from separable to bound entangled and from bound entangled to free entangled regions. We show that in any region, this state is not useful for sub-shot noise interferometry. We also relate the QFI of this state with its geometric discord and show how these two properties exhibit a similar behavior throughout this transition.

Keywords Bound entanglement · Quantum Fisher information · Quantum Discord

PACS 03.67.Mn, 03.65.Ud, 03.67.-a, 03.65.Ta

Quantum entanglement is at the heart of quantum mechanics and non-classical correlations between systems have been studied mostly from the perspective of entanglement [1]. There is still a lot to study on multipartite entanglement which appear in classes that cannot be converted to each other via LOCC [2], whereas the entanglement of bipartite systems has been understood better. There are well established measures for quantifying the entanglement of bipartite systems based on the negative eigenvalues of the partially transposed density matrix of the system [3,4]. Criteria based on such a negativity,

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i.e. negative partial transpose (NPT) implies entanglement and entanglement implies inseparability but it was shown that not all inseparable states imply NPT, i.e. there are non-separable systems with positive partial transpose (PPT) [5,6]. This kind of entanglement is called *bound entanglement* which constitutes the limits of entanglement distillation [7]. It was shown that it is possible to *activate* a three-level bipartite bound entangled state [8] and to *superactivate* a four-partite two-level state [9]. Although it is impossible to distill maximally entanglement from it, bound entanglement is surprisingly not useless for quantum information tasks: it can be used as a resource to share a secret key [10,11]. Generation and activation of bound entangled states has also been experimentally demonstrated in various settings [12–15].

On the other hand, *quantum discord* has been proposed as a new measure of non-classicality [16], such that the quantum discord $Q(\rho)$ of a bipartite state ρ on a system $H^a \otimes H^b$ with reduced density matrices ρ^a and ρ^b is given by $Q(\rho) = min_{\Pi^a}\{I(\rho) - I[\Pi^a(\rho)]\}$, where minimum of is over von Neumann measurements on party a and the state after the measurement is $\Pi^a(\rho) = \Sigma_k(\Pi^a_k \otimes I^b)\rho(\Pi^a_k \otimes I^b)$ with I^b being the identity operator on H^b [18]. The geometric discord $D(\rho)$ of a state ρ is defined as $D(\rho) = min_{\chi}||\rho - \chi||^2$, where the minimum of the square of Hilbert-Schmidt norm of the operators ρ and χ is over the set of zero-discord states χ, i.e $Q(\chi) = 0$ [17–20]. In [21], several well-known bound entangled states have been analyzed, including a special one that we will study below.

Quantum Fisher information (QFI) which characterizes the sensitivity of a quantum system with respect to the changes of a parameter of the system has been shown to be a multipartite entanglement witness: If the mean quantum Fisher information per particle of a state exceeds the so called *shot-noise limit* i.e. the ultimate limit that separable states can provide, then the state is multipartite entangled [22]. The converse is not generally true because not all pure multipartite entangled states achieve this limit, i.e. they are not *useful for sub-shot-noise interferometry* even if optimized by local operations [23]. It is also shown that the superposition of coherent spin states can surpass the shot-noise limit although they cannot surpass alone [24] and GHZ states provide the largest sensitivity, achieving the fundamental, so called Heisenberg limit [25]. Recently the quantum Fisher information has been further studied both theoretically and experimentally [26–45]. Since separability is a key issue both in QFI and in bound entanglement, a natural direction is to study the QFI of bound entangled states, especially from the point of view of usefulness for sub-shot-noise interferometry: it would be interesting to find a useful bound entangled state. Hyllus et al. studied two classes of bound entangled states, i.e. Dur states [46] and Smolin states [47,48] and showed that both of these states are not useful [49] but recently a family of bound entangled states were found to be useful [50].

In this work, we study the quantum Fisher information of the bipartite three-level state given in [8]. This state is interesting because i) it turns to be a separable, bound entangled or a free entangled state, depending on the parameter of the state, and ii) when it is bound entangled, if assisted with

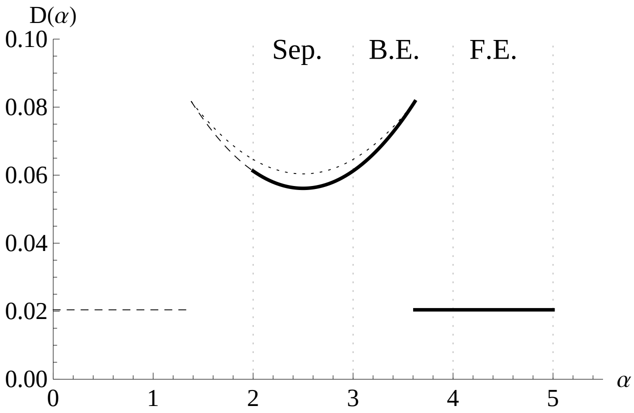

Fig. 1 (Color online) The lower bound of the geometric discord of the state for the regions of concern (the upper bound that could be derived for a specific region is given with the dotted curve) as presented in Ref.[21]. For $\alpha \geq 2$, Since the classification of the state is not known for $\alpha < 2$, we plot the bounds in that region dashed.

sufficiently many free entangled states, it can be *liberated (activated)*. The state is given in [8] as

$$\sigma_\alpha = \frac{2}{7}|\Psi_+\rangle\langle\Psi_+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \tag{1}$$

where

$$|\Psi_+\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}}, \tag{2}$$

$$\sigma_+ = \frac{|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|}{3}, \tag{3}$$

and

$$\sigma_- = \frac{|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|}{3}. \tag{4}$$

This state is separable for $2 \leq \alpha \leq 3$, bound entangled for $3 < \alpha \leq 4$ and free entangled for $4 < \alpha \leq 5$. An entanglement binding channel mapping to this state is also constructed in [51]. The geometric discord of this state is studied in [21] and it was shown that depending on the α parameter of the state, the geometric discord $D(\sigma_\alpha)$ has lower and upper bounds,

$$\frac{1}{49}(9 - 5\alpha + \alpha^2) \leq D(\sigma_\alpha) \leq \frac{1}{294}(49 - 25\alpha + 5\alpha^2). \tag{5}$$

for $\frac{5-\sqrt{5}}{2} \leq \alpha \leq \frac{5+\sqrt{5}}{2}$ and the lower bound could be found as $\frac{1}{49} \leq D(\sigma_\alpha)$ elsewhere and again this bound is not claimed to be tightest. We plot these bounds in Fig. 1: Since the state is not known to be separable or free or bound entangled for $\alpha \leq 2$, we plot that region dashed black and we are interested in the regions $2 \leq \alpha \leq 3$ (separable), $3 < \alpha \leq 4$ (bound entangled) and $4 < \alpha \leq 5$

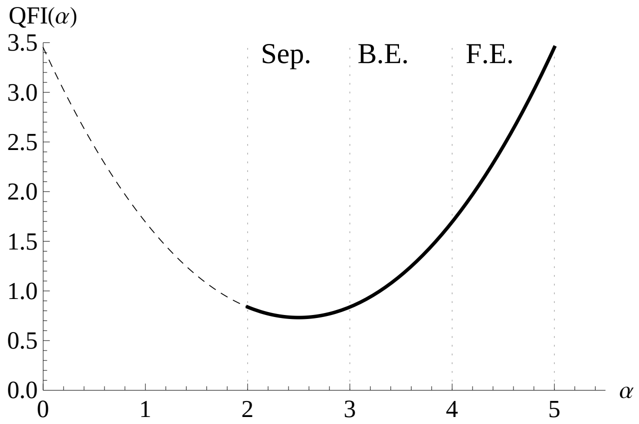

Fig. 2 (Color online) Quantum Fisher information of the state for $\alpha \geq 2$ where the state is separable, bound entangled and free entangled, respectively. Since the classification of the state is not known for $\alpha \leq 2$, we plot the bounds in that region dashed.

(free entangled). The upper bound that could be derived for a specific region is plotted with dotted curve.

The ultimate limit for the precision of the parameter ϕ of a state $\rho(\phi)$ with the unbiased estimator $\hat{\phi}$ is provided by the Cramér-Rao bound $\Delta\phi_{QCB}$, i.e. $\Delta\hat{\phi} \geq \Delta\phi_{QCB} \equiv 1/\sqrt{N_m F}$ where N_m is the number of experiments and F is the quantum Fisher information of the state. One can consider that the state acquires the parameter ϕ by a rotation operator $U_\phi = exp(i\phi J_{\overrightarrow{n}})$, i.e. $\rho(\phi) = U_\phi \rho U_\phi^\dagger$, where $J_{\overrightarrow{n}}$ is the angular momentum operator in the $\overrightarrow{n}$ direction acting on each particle. The maximal quantum Fisher information F over the directions i.e. $F = max\{F_x, F_y, F_z\}$ of a possibly mixed state ρ can be found by [26]

$$F(\rho) = c_{max} \tag{6}$$

where c_{max} is the largest eigenvalue of the symmetric matrix $\mathbf{C}$ of which elements are given as

$$\mathbf{C}_{kl} = \sum_{i \neq j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} [\langle i|J_k|j\rangle\langle j|J_l|i\rangle + \langle i|J_l|j\rangle\langle j|J_k|i\rangle], \tag{7}$$

where, $\lambda_{i,j}$ and $|i\rangle, |j\rangle$ are the eigenvalues and the associated eigenvectors of the density matrix of the state and $k, l \in \{x, y, z\}$.

A detailed analysis on the diagonal and off-diagonal elements of the $\mathbf{C}$ matrix has been provided in [23, 52, 53]. In particular, if the $\mathbf{C}$ matrix is diagonal, then the eigenvalues of the matrix are the diagonal terms and therefore the optimal direction lies at x, y or z. In the three level case, the angular momentum operators are given as:

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

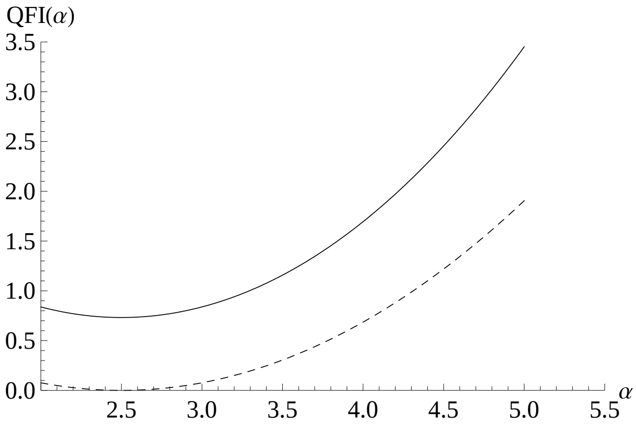

Fig. 3 (Color online) QFI of the state $\rho(\alpha)$ in x and y directions, plotted with solid and dashed lines, respectively.

We find the eigenvalues as

$$\begin{aligned} \lambda_1 &= 0.285714, \\ \lambda_{2,3,4} &= 0.238095 - 0.047619\alpha \text{ and} \\ \lambda_{5,6,7} &= 0.047619\alpha. \end{aligned}$$

Substituting these eigenvalues (and the associated eigenvectors) with the angular momentum operators to Eq.(7), the **C** matrix is easily constructed. Similar to GHZ states [26] and W states [37], the off-diagonal elements of the **C** matrix of the state σ_α are found to be zero, and the diagonal elements of the **C** matrix represent the sensitivity of the state in each direction: The two non-zero elements of the **C** matrix are found to be

$$\begin{aligned} C_{xx} &= \frac{-0.516575}{0.14966+(-0.0113379+0.00226757\alpha)\alpha} + \\ &\frac{\alpha(0.29572\ +\alpha(-0.0418673+(-0.0069107+0.00069107\alpha)\alpha))}{0.14966+(-0.0113379+0.00226757\alpha)\alpha}, \\ C_{yy} &= 1.90476 + (-1.52381 + 0.304762\alpha)\alpha. \end{aligned} \tag{9}$$

Since $C_{zz} = 0$, this state does not provide phase sensitivity in z-direction and as shown in Fig.3, the sensitivity in x-direction is always greater than the sensitivity in the y-direction. Therefore the quantum Fisher information of this state turns to be $F(\sigma_\alpha) = C_{xx}$, which we plot in Fig.2. On the contrary to the QFI of GHZ states under decoherence [26], QFI of this state does not exhibit any breaking points even at the transition points. Note that at $\alpha = 5$, $\lambda_{2,3,4}$ vanish, therefore instead of Eq.(7) provided in [26], the more recent expression for calculating the quantum Fisher information of non-full rank density matrices provided in [54–56] is more accurate.

We observe that for the region $2 \leq \alpha \leq \frac{5+\sqrt{5}}{2}$, both $F(\sigma_\alpha)$ and the lower bound of $D(\sigma_\alpha)$ exhibit the same behavior, reaching their minimum points at

$\alpha = 2.5$. For $\alpha \geq \frac{5+\sqrt{5}}{2}$ the lower bound of quantum discord could be found to be 1/49.

Our work suggests that if the QFI and the lower bound of the discord exhibit the same behavior not only for $\frac{5-\sqrt{5}}{2} \leq \alpha \leq \frac{5+\sqrt{5}}{2}$ but for values of α, then the lower bound of the discord may be $\frac{1}{49}(9 - 5\alpha + \alpha^2)$. We also find that the state σ_α is not useful for sub-shot noise interferometry, since there is an example separable state $\rho^{sep} = |11\rangle\langle 11|$ achieving $F(\rho^{sep}) = 8$ which is greater than $F(\sigma_\alpha) < 3.5$.

In conclusion, we have studied the quantum Fisher information of a bipartite three-level state which is either separable, bound entangled or free entangled state, depending only on the parameter of the state. Relating the quantum discord of the state in these regions, we have shown that the lower bound of the quantum discord and the quantum Fisher information of the state exhibit the same behavior for the region of transition from separable to bound entangled. We have also shown that quantum Fisher information of the state does not surpass that of the best separable ones, therefore this state is not useful for sub-shot noise interferometry not only when it is bound entangled but also when it is free entangled.

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